



AP Calculus AB FRQ Practice · Unit 7

A Separable Differential Equation · numberbender.com

Name: _____

Date: _____

Score: / 9

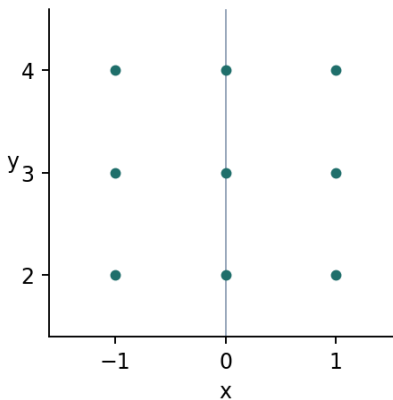
What this question practices

- Sketching a slope field
- Tangent-line approximation
- Using the second derivative to judge an approximation
- Solving a separable differential equation

No calculator is allowed for this question. Show all your work, and justify your answers. Suggested time: 15 minutes.

Consider the differential equation below. Let $y = f(x)$ be the particular solution with $f(1) = 4$.

$$\frac{dy}{dx} = 2x(y - 3)$$



(a) On the axes provided, sketch the slope field for the differential equation at the nine points indicated.

Use the axes above.

(b) Write an equation for the line tangent to the graph of f at $x = 1$, and use it to approximate $f(1.1)$.



(c) Find the second derivative in terms of x and y . Is the approximation in part (b) an overestimate or an underestimate? Explain.

(d) Find the particular solution $y = f(x)$ with $f(1) = 4$.

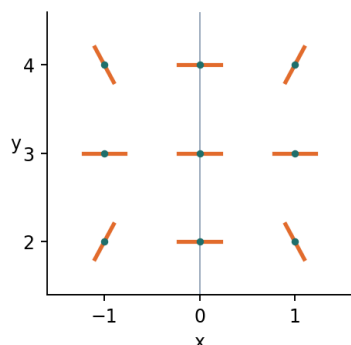




This question is worth 9 points, scored the way AP Calculus readers score free-response questions. Compare your work with the model solution, then award yourself points using the scoring notes.

(a) On the axes provided, sketch the slope field for the differential equation at the nine points indicated.

→ Along $y = 3$ and along $x = 0$ the slope is 0 (horizontal segments).



→ At $(1, 4)$ and $(-1, 2)$ the slope is 2; at $(1, 2)$ and $(-1, 4)$ the slope is -2 .

Scoring. 2 points: 1 point for zero slopes along $y = 3$ and $x = 0$, 1 point for the correct signs and relative steepness at the other four points.

(b) Write an equation for the line tangent to the graph of f at $x = 1$, and use it to approximate $f(1.1)$.

$$\frac{dy}{dx} \text{ at } (1, 4): 2(1)(4 - 3) = 2$$

$$y = 4 + 2(x - 1), \quad f(1.1) \approx 4 + 2(0.1) = 4.2$$

→ Use the point $(1, 4)$ and the slope from the differential equation.

Scoring. 2 points: 1 point for the tangent line, 1 point for the approximation 4.2.

(c) Find the second derivative in terms of x and y . Is the approximation in part (b) an overestimate or an underestimate? Explain.

$$\frac{d^2y}{dx^2} = 2(y - 3) + 2x \frac{dy}{dx} = 2(y - 3) + 4x^2(y - 3) = 2(y - 3)(1 + 2x^2)$$

$$\frac{d^2y}{dx^2} \text{ at } (1, 4): 2(1)(3) = 6 > 0$$

→ The graph of f is concave up near $x = 1$, so the tangent line lies below the curve and 4.2 is an **underestimate**.

Scoring. 2 points: 1 point for the second derivative, 1 point for underestimate with the concavity reason.



(d) Find the particular solution $y = f(x)$ with $f(1) = 4$.

→ Separate the variables, integrate both sides, and use $f(1) = 4$ to find the constant.

$$\frac{dy}{y-3} = 2x \, dx \Rightarrow \ln|y-3| = x^2 + C$$

$$\ln|4-3| = 1 + C \Rightarrow C = -1$$

$$y - 3 = e^{x^2 - 1} \Rightarrow f(x) = 3 + e^{x^2 - 1}$$

→ Near the initial condition $y - 3 > 0$, so the absolute value can be dropped. (Check: $f(1.1) \approx 4.234$, slightly above the estimate in part (b).)

Scoring. 3 points: 1 point for separating variables, 1 point for the antiderivatives, 1 point for using the initial condition to find the constant and solving for y .

