MATH 160 — Graph Theory

numberbender.com

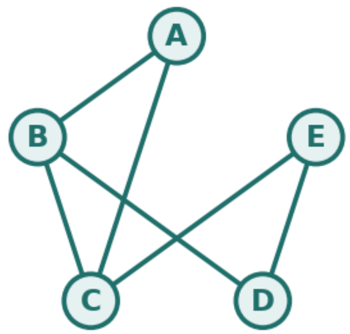
Name: ______________________________ Date: _______________ Score: / 10

Learning Objectives

- Compute vertex degrees and apply the Handshaking Theorem
- Identify Euler circuits/paths and Hamiltonian paths
- Write adjacency matrices and interpret graph structure
- Apply Euler's formula V – E + F = 2 for planar graphs

For every graph problem: first find all vertex degrees. For Euler circuits: check that ALL vertices have even degree.

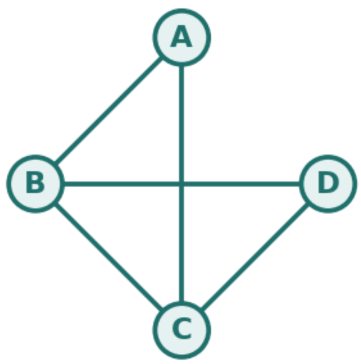
1. Find the degree of each vertex and verify the Handshaking Theorem (sum of degrees = 2|E|).



Find deg(A), deg(B), deg(C), deg(D), deg(E)

Answer: ________________________________

2. Does the graph have an Euler circuit? An Euler path? State the condition and check.

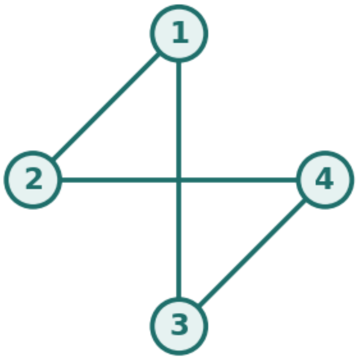


Does an Euler circuit exist?

Answer: ________________________________

Scan to watch

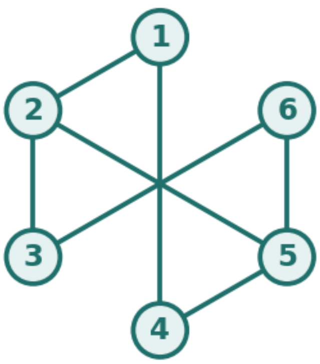
3. Write the adjacency matrix for the graph with vertices {1, 2, 3, 4} and edges {(1,2),(1,3),(2,4),(3,4)}.



Write the 4×4 adjacency matrix.

Answer: ______________________________

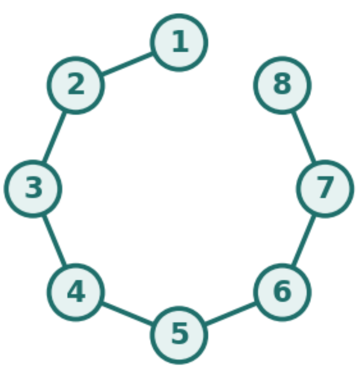
4. Is the graph bipartite? If yes, find the two vertex groups.



Color vertices using 2 colors — bipartite?

Answer: ______________________________

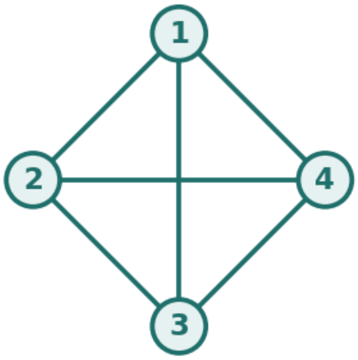
5. A connected graph has 8 vertices and is a tree. How many edges does it have? Draw a possible spanning tree.



Tree with 8 vertices — how many edges?

Answer: ______________________________

6. What is the chromatic number χ(G) of a complete graph K■? Color it using the minimum number of colors.

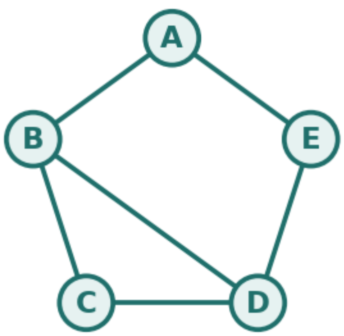


K_4 — complete graph on 4 vertices

Answer: ______________________________________

---

7. Find a Hamiltonian path in the graph (a path that visits every vertex exactly once). Does a Hamiltonian circuit exist?



Find a Hamiltonian path.

Answer: ______________________________________

---

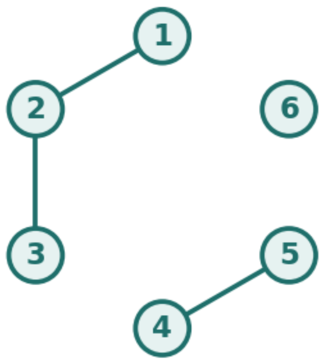
8. A weighted graph has edges: A–B(3), A–C(5), B–C(1), B–D(4), C–D(2). Find the shortest path from A to D.

$$A \xrightarrow{3} B \xrightarrow{4} D, \quad A \xrightarrow{3} B \xrightarrow{1} C \xrightarrow{2} D$$

Answer: ______________________________________

Scan to watch

9. How many connected components does the graph have? List the vertices in each component.



How many connected components?

Answer: ______________________________

---

10. A connected planar graph has 10 vertices and 15 edges. How many faces does it have? (Use Euler's formula: V – E + F = 2)

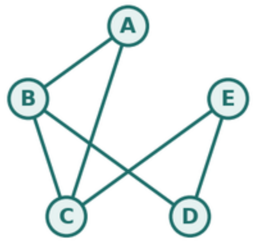
$$V - E + F = 2, \quad V = 10, \; E = 15$$

Answer: ______________________________

Problems 1–2: degree sum and Euler circuit condition — foundational. Problem 3 (adjacency matrix): connect to matrix concepts from Linear Algebra. Problem 10 (Euler's formula): make sure students count the outer face.

Solutions

1. Find the degree of each vertex and verify the Handshaking Theorem (sum of degrees = 2|E|).

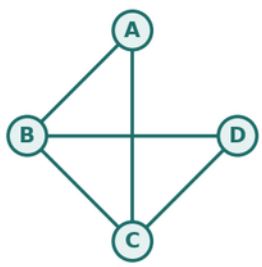


Find deg(A), deg(B), deg(C), deg(D), deg(E)

→ *deg(A)=2, deg(B)=3, deg(C)=3, deg(D)=2, deg(E)=2.*
→ *Sum = 2+3+3+2+2 = 12.*
→ *Edges: (A,B),(A,C),(B,C),(B,D),(C,E),(D,E) → |E|=6.*
→ *Handshaking: sum of degrees = 2|E| = 2·6 = 12. ✓*

Answer: $\sum \deg = 12 = 2 \times 6$

2. Does the graph have an Euler circuit? An Euler path? State the condition and check.



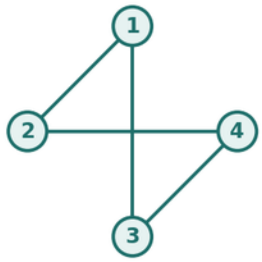
Does an Euler circuit exist?

→ *Euler circuit exists iff ALL vertices have even degree.*
→ *deg(A)=2, deg(B)=3, deg(C)=3, deg(D)=2.*
→ *B and C have odd degree → no Euler circuit.*
→ *Euler path exists iff exactly 0 or 2 vertices have odd degree — here exactly 2 (B and C). ✓*
→ *Euler path starts at B and ends at C (or vice versa).*

Answer: No Euler circuit (odd-degree vertices); Euler path exists

Scan to watch

3. Write the adjacency matrix for the graph with vertices {1, 2, 3, 4} and edges {(1,2),(1,3),(2,4),(3,4)}.



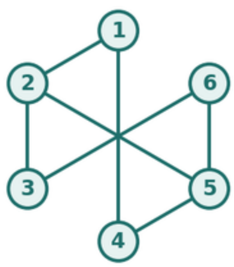
Write the 4×4 adjacency matrix.

→ *Row/col order: 1,2,3,4.*
→ *Row 1: (1,1)=0,(1,2)=1,(1,3)=1,(1,4)=0.*
→ *Row 2: 1,0,0,1. Row 3: 1,0,0,1. Row 4: 0,1,1,0.*
→ *Adjacency matrix is symmetric for undirected graphs.*

Answer: $A_{ij} = 1 \text{ if } (i, j) \in E, \ 0 \text{ otherwise}$

---

4. Is the graph bipartite? If yes, find the two vertex groups.



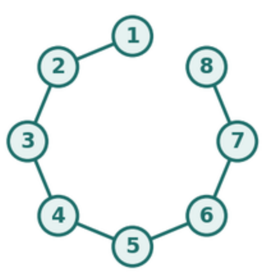
Color vertices using 2 colors — bipartite?

→ *A graph is bipartite iff it contains NO odd-length cycle.*
→ *Try 2-coloring starting from vertex 1: color alternating.*
→ *Look for any triangle (3-cycle) or odd cycle — if found, NOT bipartite.*
→ *With 6 vertices and these edges, trace cycles to check.*

Answer: Check: odd cycle $\Rightarrow$ not bipartite

---

5. A connected graph has 8 vertices and is a tree. How many edges does it have? Draw a possible spanning tree.



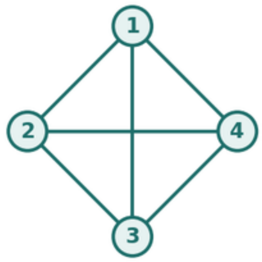
Tree with 8 vertices — how many edges?

→ *A tree with n vertices has exactly n–1 edges.*
→ *8 vertices → 7 edges.*
→ *A tree is connected and acyclic (no cycles).*
→ *The drawing shows a path graph — one example of a tree on 8 vertices.*

Answer: $|E| = n - 1 = 8 - 1 = 7 \text{ edges}$

---

6. What is the chromatic number χ(G) of a complete graph K■? Color it using the minimum number of colors.



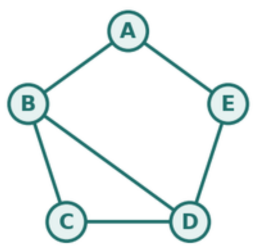
K_4 — complete graph on 4 vertices

→ *K■: every vertex is adjacent to every other vertex.*
→ *No two adjacent vertices can share a color.*
→ *All 4 vertices are mutually adjacent → need 4 different colors.*
→ *χ(K■) = n for complete graphs.*

Answer: $\chi(K_4) = 4$

---

7. Find a Hamiltonian path in the graph (a path that visits every vertex exactly once). Does a Hamiltonian circuit exist?



Find a Hamiltonian path.

→ *Hamiltonian path: visit each vertex exactly once.*
→ *Try: A→B→C→D→E — check if all edges exist: (A,B)✓,(B,C)✓,(C,D)✓,(D,E)✓.*
→ *Hamiltonian circuit: also need edge (E,A) → (E,A)=(4,0) ✓.*
→ *Hamiltonian circuit: A→B→C→D→E→A.*

Answer: $A \to B \to C \to D \to E \to A \text{ (circuit exists)}$

---

8. A weighted graph has edges: A–B(3), A–C(5), B–C(1), B–D(4), C–D(2). Find the shortest path from A to D.

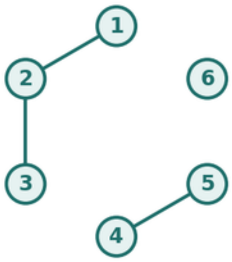
$A \xrightarrow{3} B \xrightarrow{4} D, \quad A \xrightarrow{3} B \xrightarrow{1} C \xrightarrow{2} D$

→ *Option 1: A→B→D = 3+4 = 7.*
→ *Option 2: A→C→D = 5+2 = 7.*
→ *Option 3: A→B→C→D = 3+1+2 = 6. ← shortest*
→ *Shortest path: A→B→C→D with total weight 6.*

Answer: $A \to B \to C \to D, \ \text{cost} = 3 + 1 + 2 = 6$

---

9. How many connected components does the graph have? List the vertices in each component.



How many connected components?

→ *Component 1: vertices 1–2–3 connected via edges (1,2),(2,3).*
→ *Component 2: vertices 4–5 connected via edge (4,5).*
→ *Component 3: vertex 6 is isolated (no edges).*
→ *3 connected components total.*

Answer: $3 \text{ components: } \{1,2,3\}, \{4,5\}, \{6\}$

10. A connected planar graph has 10 vertices and 15 edges. How many faces does it have? (Use Euler's formula: V – E + F = 2)

$V - E + F = 2, \quad V = 10,\ E = 15$

→ *Euler's formula for connected planar graphs: V – E + F = 2.*
→ *F = 2 – V + E = 2 – 10 + 15 = 7.*
→ *One of the 7 faces is the outer (unbounded) face.*

Answer: $F = 2 - V + E = 2 - 10 + 15 = 7 \text{ faces}$

Scan to watch