MATH 210 — Applications of Derivatives

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Learning Objectives

- Find critical numbers and classify local extrema
- Determine intervals of increase, decrease, and concavity
- Solve optimization problems using derivatives
- Apply related rates and the Mean Value Theorem

For optimization: define variables, write the objective function, apply constraints, then differentiate and solve.

1. Find all critical numbers of f(x). Then classify each as a local max, min, or neither.

$$f(x) = x^3 - 3x^2 - 9x + 5$$

Answer: ________________________________

2. Find the intervals where f(x) is increasing and decreasing.

$$f(x) = 2x^3 - 12x + 1$$

Answer: ________________________________

3. Find the absolute maximum and minimum of f(x) on the closed interval [0, 4].

$$f(x) = x^3 - 6x^2 + 8 \quad \text{on } [0, 4]$$

Answer: ________________________________

4. Determine the concavity of f(x) and find all inflection points.

$$f(x) = x^4 - 4x^3$$

Answer: ________________________________

5. Optimization: A farmer has 200 m of fencing to enclose a rectangular plot. Find the dimensions that maximize the area.

$$A = x(100 - x), \quad 0 \le x \le 100$$

Answer: ________________________________

6. Related Rates: A ladder 10 m long leans against a wall. The bottom slides away at 2 m/s. How fast is the top sliding down when the bottom is 6 m from the wall?

$$x^2 + y^2 = 100, \quad \frac{dx}{dt} = 2$$

Answer: ________________________________

7. Use the Mean Value Theorem. Find c in (1, 3) satisfying the MVT conclusion.

$$f(x) = x^2 - 2x \quad \text{on } [1, 3]$$

Answer: ______________________________________

8. Apply L'Hôpital's Rule to evaluate the limit.

$$\lim_{x \to 0} \frac{e^x - 1}{x}$$

Answer: ______________________________________

9. Find the linear approximation (tangent line approximation) of f(x) at x = 0.

$$f(x) = \sqrt{1 + x} \quad \text{at } a = 0$$

Answer: ______________________________________

10. Optimization: A box with a square base and open top must hold 32 cm³. Find dimensions minimizing surface area.

$$S = x^2 + \frac{4 \cdot 32}{x}, \quad V = x^2 h = 32$$

Answer: ______________________________________

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ANSWER KEY & SOLUTIONS

Problems 5 and 10 are optimization — classic exam problems. Problem 6 (related rates): emphasize implicit differentiation with respect to t. Problem 8 (L'Hôpital's): verify 0/0 indeterminate form first.

Solutions

1. Find all critical numbers of f(x). Then classify each as a local max, min, or neither.

$f(x) = x^3 - 3x^2 - 9x + 5$

→ *f'(x) = $3x^2$ – 6x – 9 = 3(x^2–2x–3) = 3(x–3)(x+1).*
→ *Critical numbers: x = –1 and x = 3.*
→ *f''(x) = 6x – 6.*
→ *f''(–1) = –12 < 0 → local max. f''(3) = 12 > 0 → local min.*

Answer: $x = -1 \text{ (local max)},\ x = 3 \text{ (local min)}$

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2. Find the intervals where f(x) is increasing and decreasing.

$f(x) = 2x^3 - 12x + 1$

→ *f'(x) = $6x^2$ – 12 = 6(x^2–2).*
→ *f'(x) = 0: x = ±√2.*
→ *f'(x) > 0 (increasing) when |x| > √2.*
→ *f'(x) < 0 (decreasing) on (–√2, √2).*

Answer: $\text{Increasing on } (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

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3. Find the absolute maximum and minimum of f(x) on the closed interval [0, 4].

$f(x) = x^3 - 6x^2 + 8 \quad \text{on } [0, 4]$

→ *f'(x) = $3x^2$ – 12x = 3x(x–4).*
→ *Critical points in [0,4]: x = 0 and x = 4 (endpoints).*
→ *Also x = 0 is a critical number (f'(0)=0) — it's an endpoint.*
→ *f(0) = 8, f(4) = 64 – 96 + 8 = –24.*
→ *Also check x = 4 is endpoint. Abs max = 8, abs min = –24.*

Answer: $\text{Abs max: } f(0) = 8; \quad \text{Abs min: } f(4) = -24$

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4. Determine the concavity of f(x) and find all inflection points.

$f(x) = x^4 - 4x^3$

→ *f'(x) = $4x^3$ – $12x^2$.*
→ *f''(x) = $12x^2$ – 24x = 12x(x–2).*
→ *f''(x) = 0 at x = 0 and x = 2.*
→ *f''(x) changes sign at both — inflection points at x = 0 and x = 2.*

Answer: $\text{Inflection at } x = 0 \text{ and } x = 2$

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5. Optimization: A farmer has 200 m of fencing to enclose a rectangular plot. Find the dimensions that maximize the area.

$$A = x(100 - x), \quad 0 \le x \le 100$$

→ *Perimeter constraint: 2x + 2y = 200, so y = 100 – x.*
→ *Area: A(x) = x(100 – x) = 100x –* x^2*.*
→ *A′(x) = 100 – 2x = 0 → x = 50.*
→ *Both sides equal 50 m. Max area = 50·50 = 2500* m^2*.*

Answer: $x = 50 \text{ m}, \; y = 50 \text{ m}, \; A_{max} = 2500 \text{ m}^2$

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6. Related Rates: A ladder 10 m long leans against a wall. The bottom slides away at 2 m/s. How fast is the top sliding down when the bottom is 6 m from the wall?

$$x^2 + y^2 = 100, \quad \frac{dx}{dt} = 2$$

→ *Differentiate* $x^2 + y^2 = 100$ *with respect to t.*
→ *2x(dx/dt) + 2y(dy/dt) = 0.*
→ *When x = 6: y = √(100–36) = 8.*
→ *2(6)(2) + 2(8)(dy/dt) = 0 → dy/dt = –24/16 = –3/2 m/s.*

Answer: $\frac{dy}{dt} = -\frac{3}{2} \text{ m/s}$

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7. Use the Mean Value Theorem. Find c in (1, 3) satisfying the MVT conclusion.

$$f(x) = x^2 - 2x \quad \text{on } [1, 3]$$

→ *MVT: f′(c) = [f(3)–f(1)]/(3–1).*
→ *f(3) = 9–6 = 3. f(1) = 1–2 = –1.*
→ *Average rate: (3–(–1))/2 = 2.*
→ *f′(x) = 2x–2 = 2 → x = 2. So c = 2 ∈ (1,3).* ✓

Answer: $c = 2$

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8. Apply L'Hôpital's Rule to evaluate the limit.

$$\lim_{x \to 0} \frac{e^x - 1}{x}$$

→ *Form is 0/0 at x = 0, so L'Hôpital's Rule applies.*
→ *Differentiate numerator: d/dx[*e^x*–1] =* e^x*.*
→ *Differentiate denominator: d/dx[x] = 1.*
→ *Limit = lim(*e^x*/1) =* e^0 *= 1.*

Answer: $\lim_{x \to 0} \frac{e^x}{1} = 1$

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9. Find the linear approximation (tangent line approximation) of f(x) at x = 0.

$$f(x) = \sqrt{1 + x} \quad \text{at } a = 0$$

→ *L(x) = f(a) + f′(a)(x – a).*
→ *f(0) = 1. f′(x) = 1/(2√(1+x)) → f′(0) = 1/2.*
→ *L(x) = 1 + (1/2)(x – 0) = 1 + x/2.*

Answer: $L(x) = 1 + \frac{x}{2}$

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10. Optimization: A box with a square base and open top must hold 32 cm³. Find dimensions minimizing surface area.

$$S = x^2 + \frac{4 \cdot 32}{x}, \quad V = x^2 h = 32$$

→ *Volume: $x^2h = 32$, so $h = 32/x^2$.*
→ *Surface area (open top): $S = x^2 + 4xh = x^2 + 128/x$.*
→ *$S'(x) = 2x - 128/x^2 = 0 \to 2x^3 = 128 \to x = 4$.*
→ *$h = 32/16 = 2$. Box: 4 cm × 4 cm base, 2 cm tall.*

Answer: $x = 4\text{ cm},\ h = 2\text{ cm}$

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