MATH 220 — Applications of Integration

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Learning Objectives

- Find the area between two curves using integration
- Compute the average value of a function over an interval
- Set up and evaluate volumes of revolution using the Disk Method
- Apply the Washer Method for solids with holes

For area between curves, always identify which function is on top. For volumes, sketch the region and identify the radius function.

1. Find the area between f(x) = x² and g(x) = x from x = 0 to x = 1.

$$\int_0^1 (x - x^2)dx$$

Answer: ________________________________

2. Find the area enclosed between f(x) = 4 – x² and g(x) = x + 2. (Find intersection points first.)

$$\int_{-2}^{1} [(4 - x^2) - (x + 2)]dx = \int_{-2}^{1} (2 - x - x^2)\,dx$$

Answer: ________________________________

3. Find the area between f(x) = x³ and g(x) = x on [0, 1].

$$\int_0^1 (x - x^3)dx$$

Answer: ________________________________

4. Find the average value of f(x) = x² – 2x + 3 on [0, 4].

$$\bar{f} = \frac{1}{4}\int_0^4 (x^2 - 2x + 3)\,dx$$

Answer: ________________________________

5. Find the average value of f(x) = 3x² on [1, 3].

$$\bar{f} = \frac{1}{2}\int_1^3 3x^2\,dx$$

Answer: ________________________________

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6. Use the Disk Method to find the volume of the solid formed by rotating y = √x about the x-axis from x = 0 to x = 4.

$$V = \pi \int_0^4 (\sqrt{x})^2 dx = \pi \int_0^4 x\, dx$$

Answer: ______________________________

7. Use the Disk Method. Find the volume when $y = x^2$ is rotated about the x-axis from x = 0 to x = 2.

$$V = \pi \int_0^2 x^4\, dx$$

Answer: ______________________________

8. Use the Washer Method. Find the volume when the region between y = x and $y = x^2$ (on [0,1]) is rotated about the x-axis.

$$V = \pi \int_0^1 [x^2 - (x^2)^2]\, dx = \pi \int_0^1 (x^2 - x^4)\, dx$$

Answer: ______________________________

9. Find the area of the region enclosed by $f(x) = 6x - x^2$ and $g(x) = x^2$.

$$\int_0^3 [(6x - x^2) - x^2]\, dx = \int_0^3 (6x - 2x^2)\, dx$$

Answer: ______________________________

10. Find the volume when y = 1/x is rotated about the x-axis from x = 1 to x = 3 (Disk Method).

$$V = \pi \int_1^3 \frac{1}{x^2}\, dx$$

Answer: ______________________________

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Area problems (1–3, 9): students must verify which curve is on top before integrating. Problem 2 requires finding intersections first — a common exam task.

Solutions

1. Find the area between $f(x) = x^2$ and $g(x) = x$ from $x = 0$ to $x = 1$.

$$\int_0^1 (x - x^2)dx$$

→ *Check: g(x) = x > f(x) = x^2 on (0, 1), so integrate g – f.*
→ *Antiderivative: $x^2/2 - x^3/3$.*
→ *At x = 1: 1/2 – 1/3 = 1/6. At x = 0: 0.*
→ *Area = 1/6.*

Answer: $$\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$$

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2. Find the area enclosed between $f(x) = 4 - x^2$ and $g(x) = x + 2$. (Find intersection points first.)

$$\int_{-2}^1 [(4 - x^2) - (x + 2)]dx = \int_{-2}^1 (2 - x - x^2)\,dx$$

→ *Set equal: $4 - x^2 = x + 2$ → $x^2 + x - 2 = 0$ → (x+2)(x–1) = 0.*
→ *Intersections at x = –2 and x = 1.*
→ *On [–2, 1]: f(x) > g(x), so integrate $(4 - x^2) - (x + 2) = 2 - x - x^2$.*
→ *Antiderivative: $2x - x^2/2 - x^3/3$.*
→ *At x=1: 2 – 1/2 – 1/3 = 7/6. At x=–2: –4 – 2 + 8/3 = –10/3.*
→ *Area = 7/6 – (–10/3) = 7/6 + 20/6 = 27/6 = 9/2.*

Answer: $$\frac{9}{2}$$

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3. Find the area between $f(x) = x^3$ and $g(x) = x$ on [0, 1].

$$\int_0^1 (x - x^3)dx$$

→ *On [0, 1]: x > x^3, so integrate $x - x^3$.*
→ *Antiderivative: $x^2/2 - x^4/4$.*
→ *At x = 1: 1/2 – 1/4 = 1/4. At x = 0: 0.*
→ *Area = 1/4.*

Answer: $$\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{1}{4}$$

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4. Find the average value of $f(x) = x^2 – 2x + 3$ on [0, 4].

$$\bar{f} = \frac{1}{4}\int_0^4 (x^2 - 2x + 3)\,dx$$

- → *Average value formula: f_avg = 1/(b–a) · ∫■■ f(x) dx.*
- → *Antiderivative: $x^3/3 – x^2 + 3x$.*
- → *At x = 4: 64/3 – 16 + 12 = 64/3 – 4 = 52/3.*
- → *At x = 0: 0.*
- → *f_avg = (1/4)(52/3) = 52/12 = 13/3.*

Answer: $$\frac{1}{4}\cdot\frac{28}{3} = \frac{7}{3}$$

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5. Find the average value of $f(x) = 3x^2$ on [1, 3].

$$\bar{f} = \frac{1}{2}\int_1^3 3x^2\,dx$$

- → *Average value: 1/(3–1) · ∫■³ $3x^2$ dx.*
- → *Antiderivative of $3x^2$ is x^3.*
- → *Evaluate: $[x^3]$■³ = 27 – 1 = 26.*
- → *Average = (1/2)(26) = 13.*

Answer: $$\frac{1}{2}[x^3]_1^3 = \frac{26}{2} = 13$$

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6. Use the Disk Method to find the volume of the solid formed by rotating y = √x about the x-axis from x = 0 to x = 4.

$$V = \pi\int_0^4 (\sqrt{x})^2 dx = \pi\int_0^4 x\,dx$$

- → *Disk Method: V = π ∫■■ $[R(x)]^2$ dx.*
- → *R(x) = √x, so $[R(x)]^2 = x$.*
- → *V = π ∫■■ x dx = π $[x^2/2]$■■ = π(8 – 0) = 8π.*

Answer: $$\pi\left[\frac{x^2}{2}\right]_0^4 = 8\pi$$

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7. Use the Disk Method. Find the volume when $y = x^2$ is rotated about the x-axis from x = 0 to x = 2.

$$V = \pi\int_0^2 x^4\,dx$$

- → *Disk Method: V = π ∫■² $(x^2)^2$ dx = π ∫■² x^4 dx.*
- → *Antiderivative: $x^5/5$.*
- → *V = π $[(2)^5/5 – 0]$ = 32π/5.*

Answer: $$\pi\left[\frac{x^5}{5}\right]_0^2 = \frac{32\pi}{5}$$

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8. Use the Washer Method. Find the volume when the region between y = x and $y = x^2$ (on [0,1]) is rotated about the x-axis.

$$V = \pi\int_0^1 [x^2 - (x^2)^2]dx = \pi\int_0^1 (x^2 - x^4)\,dx$$

- → *Washer Method: V = π ∫ $[R(x)^2 – r(x)^2]$ dx.*
- → *Outer R(x) = x, inner r(x) = x^2.*
- → *V = π ∫■¹ $(x^2 – x^4)$ dx = π $[x^3/3 – x^5/5]$■¹.*
- → *V = π(1/3 – 1/5) = π(2/15) = 2π/15.*

Answer: $$\pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{2\pi}{15}$$

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9. Find the area of the region enclosed by $f(x) = 6x - x^2$ and $g(x) = x^2$.

$$\int_0^3 [(6x - x^2) - x^2]dx = \int_0^3 (6x - 2x^2)\,dx$$

→ *Set equal: $6x - x^2 = x^2$ → $6x = 2x^2$ → $x = 0$ or $x = 3$.*
→ *On [0, 3]: $6x - x^2$ > x^2, so integrate $(6x - 2x^2)$.*
→ *Antiderivative: $3x^2 - 2x^3/3$.*
→ *At x = 3: 27 – 18 = 9. At x = 0: 0. Area = 9.*

Answer: $\left[3x^2 - \frac{2x^3}{3}\right]_0^3 = 9$

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10. Find the volume when y = 1/x is rotated about the x-axis from x = 1 to x = 3 (Disk Method).

$$V = \pi \int_1^3 \frac{1}{x^2}dx$$

→ *Disk Method: V = π ∫■³ $(1/x)^2$ dx = π ∫■³ x^{-2} dx.*
→ *Antiderivative of x^{-2} is $-x^{-1}$.*
→ *V = π [–1/x]■³ = π(–1/3 – (–1)) = π(2/3) = 2π/3.*

Answer: $\pi\left[-\frac{1}{x}\right]_1^3 = \frac{2\pi}{3}$