MATH 220 — Differential Equations

numberbender.com

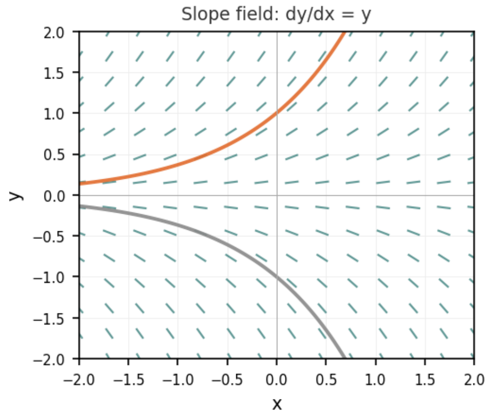
Name: ______________________________ Date: _______________ Score: / 10

Learning Objectives

- Read and interpret slope fields; sketch solution curves through given points
- Solve separable differential equations with initial conditions
- Model exponential growth and decay by setting up and solving dP/dt = kP
- Apply Newton's Law of Cooling and the logistic growth model

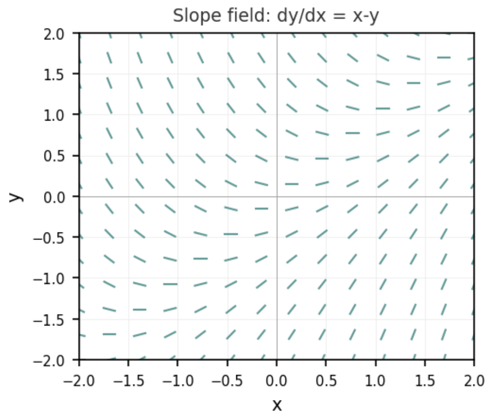
For separable DEs: separate variables completely before integrating. Always apply the initial condition AFTER finding the general solution.

1. The slope field below is for dy/dx = y. Sketch the solution curve through (0, 1). What is the general solution?



Answer: ______________________________

2. Analyze the slope field below (dy/dx = x – y). Describe where slopes are zero, positive, and negative.



Answer: ______________________________

Scan to watch

3. Solve the separable differential equation with initial condition.

$$\frac{dy}{dx} = 3x^2 y, \quad y(0) = 2$$

Answer: ______________________________

4. Solve the separable differential equation.

$$\frac{dy}{dx} = \frac{x^2}{y^2}, \quad y(0) = 1$$

Answer: ______________________________

5. Solve the separable DE.

$$\frac{dy}{dx} = e^{x-y} = \frac{e^x}{e^y}$$

Answer: ______________________________

6. A bacterial population grows at a rate proportional to its size. Write and solve the differential equation. The population doubles in 3 hours, starting at P0 = 500.

$$\frac{dP}{dt} = kP, \quad P(0) = 500, \quad P(3) = 1000$$

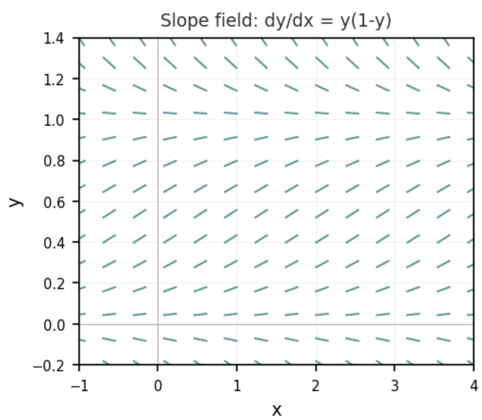
Answer: ______________________________

7. Newton's Law of Cooling: dT/dt = –k(T – T■). A cup of coffee at 90°C cools in a 20°C room. After 5 minutes, it is 70°C. Find T(t) and when it reaches 40°C.

$$\frac{dT}{dt} = -k(T - 20), \quad T(0) = 90, \quad T(5) = 70$$

Answer: ______________________________

8. The logistic growth model is dP/dt = rP(1 – P/K). With r = 1 and K = 1, the slope field is shown. Identify equilibrium solutions and describe population behavior.



Answer: ______________________________

9. Solve the first-order linear differential equation.

$$\frac{dy}{dx} + 2y = 4x, \quad y(0) = 1$$

Answer: ____________________________________

---

10. Verify that y = x² + Ce^(–x) is a solution to dy/dx + y = 2x + x².

$$\frac{dy}{dx} + y = 2x + x^2$$

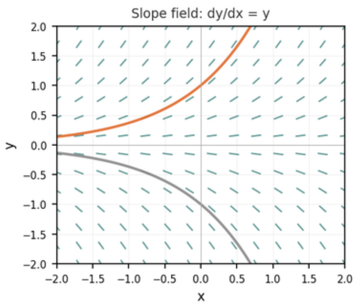
Answer: ____________________________________

Scan to watch

Problems 1–2 are slope field interpretation — build visual intuition before algebra. Problem 6 (bacterial growth): emphasize writing the DE first, then solving. Problem 8 (logistic): equilibrium analysis is more important than the explicit solution.

Solutions

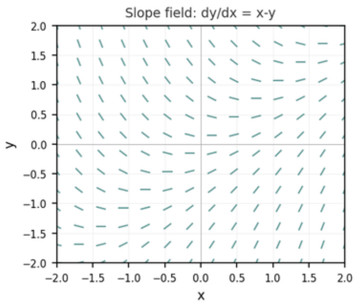
1. The slope field below is for dy/dx = y. Sketch the solution curve through (0, 1). What is the general solution?



→ *dy/dx = y is a separable equation: dy/y = dx.*
→ *Integrate both sides: ln|y| = x + C■ → y = Ce^x.*
→ *Initial condition y(0)=1: 1=Ce^0=C → C=1.*
→ *Solution: y = e^x. The orange curve in the slope field.*

Answer: $y = Ce^x, \quad \text{through } (0,1): \; y = e^x$

2. Analyze the slope field below (dy/dx = x – y). Describe where slopes are zero, positive, and negative.



→ *Slopes are zero (dy/dx=0) when x=y — along the line y=x.*
→ *Slopes are positive when x > y (above the line y=x: dy/dx>0).*
→ *Slopes are negative when x < y.*
→ *This is a first-order linear DE. Integrating factor: e^x.*
→ *General solution: y = x – 1 + Ce^{-x}.*

Answer: $\text{Nullcline: } x = y \text{ (zero slope along } y = x)$

3. Solve the separable differential equation with initial condition.

$$\frac{dy}{dx} = 3x^2 y, \quad y(0) = 2$$

→ *Separate: dy/y = $3x^2$ dx.*
→ *Integrate: ln|y| = x^3 + C.*
→ *y = Ae^{x^3}. Apply y(0)=2: A=2.*
→ *Solution: y = $2e^{x^3}$.*

Answer: $y = 2e^{x^3}$

---

4. Solve the separable differential equation.

$$\frac{dy}{dx} = \frac{x^2}{y^2}, \quad y(0) = 1$$

→ *Separate: y^2 dy = x^2 dx.*
→ *Integrate: $y^3/3$ = $x^3/3$ + C.*
→ *y^3 = x^3 + K. Apply y(0)=1: 1=K.*
→ *y^3 = x^3+1 → y = $(x^3+1)^{1/3}$.*

Answer: $y = (x^3 + 1)^{1/3}$

---

5. Solve the separable DE.

$$\frac{dy}{dx} = e^{x-y} = \frac{e^x}{e^y}$$

→ *Separate: e^y dy = e^x dx.*
→ *Integrate: e^y = e^x + C.*
→ *y = ln(e^x + C).*

Answer: $e^y = e^x + C \Rightarrow y = \ln(e^x + C)$

---

6. A bacterial population grows at a rate proportional to its size. Write and solve the differential equation. The population doubles in 3 hours, starting at P0 = 500.

$$\frac{dP}{dt} = kP, \quad P(0) = 500, \quad P(3) = 1000$$

→ *Separable: dP/P = k dt -> ln P = kt + C -> P = P0·e^{kt}.*
→ *P(0)=500: P■=500.*
→ *P(3)=1000: $500e^{3k}$=1000 → e^{3k}=2 → k=ln2/3.*
→ *P(t) = 500·$e^{(ln2/3)t}$ = 500·$2^{t/3}$.*
→ *Find P(9): 500·2^3 = 4000 bacteria.*

Answer: $P(t) = 500\,e^{(\ln 2/3)\,t}$

---

7. Newton's Law of Cooling: dT/dt = –k(T – T■). A cup of coffee at 90°C cools in a 20°C room. After 5 minutes, it is 70°C. Find T(t) and when it reaches 40°C.

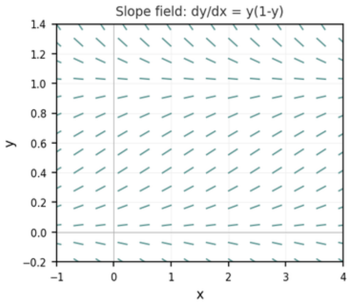
$$\frac{dT}{dt} = -k(T - 20), \quad T(0) = 90, \quad T(5) = 70$$

→ *Let u=T–20. Then du/dt=–ku → u=u■e^{-kt}.*
→ *T(t) = 20+$70e^{-kt}$. At t=5: 70=$70e^{-5k}$... wait.*
→ *T(5)=70 → 70=20+$70e^{-5k}$ → 50=$70e^{-5k}$ → e^{-5k}=5/7.*
→ *k=–(1/5)ln(5/7)=(1/5)ln(7/5).*
→ *T=40: 40=20+$70e^{-kt}$ → e^{-kt}=2/7 → t=ln(7/2)/k.*

Answer: $T(t) = 20 + 70e^{-kt}, \quad k = \frac{1}{5}\ln\frac{7}{5}$

---

8. The logistic growth model is dP/dt = rP(1 – P/K). With r = 1 and K = 1, the slope field is shown. Identify equilibrium solutions and describe population behavior.



→ *Set dP/dt=0: P(1–P)=0 → P=0 or P=1.*
→ *P=0 is an unstable equilibrium (populations grow away from 0).*
→ *P=1 (carrying capacity K=1) is a stable equilibrium.*
→ *Populations below K increase toward K; populations above K decrease toward K.*
→ *S-shaped (sigmoidal) solution curves are visible in the slope field.*

Answer: $P = 0 \text{ (unstable)},\ P = 1 \text{ (stable carrying capacity)}$

---

9. Solve the first-order linear differential equation.

$$\frac{dy}{dx} + 2y = 4x, \quad y(0) = 1$$

→ *Standard form: dy/dx + P(x)y = Q(x). Here P=2, Q=4x.*
→ *Integrating factor:* $\mu = e^{\int 2dx} = e^{2x}$.
→ *Multiply:* $d/dx[e^{2x}y] = 4xe^{2x}$.
→ *Integrate RHS by parts:* $\int 4xe^{2x}dx = 2xe^{2x} - e^{2x} + C$.
→ $e^{2x}y = 2xe^{2x} - e^{2x} + C \rightarrow y = 2x - 1 + Ce^{-2x}$.
→ *y(0)=1: 1=0–1+C → C=2.* $y = 2x - 1 + 2e^{-2x}$.

Answer: $y = 2x - 1 + 2e^{-2x}$

---

10. Verify that y = x² + Ce^(–x) is a solution to dy/dx + y = 2x + x².

$$\frac{dy}{dx} + y = 2x + x^2$$

→ *Compute dy/dx:* $dy/dx = 2x - Ce^{-x}$.
→ $LHS = dy/dx + y = (2x - Ce^{-x}) + (x^2 + Ce^{-x})$.
→ $= 2x - Ce^{-x} + x^2 + Ce^{-x} = 2x + x^2$.
→ $LHS = RHS = 2x + x^2$. ✓ $y = x^2 + Ce^{-x}$ *is the general solution.*

Answer: $\text{Verified: LHS} = 2x + x^2 = \text{RHS}$

Scan to watch