MATH 250 — Systems & Row Reduction

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Learning Objectives

- Write a linear system as an augmented matrix [A | b]
- Apply Gaussian elimination and back-substitution
- Identify consistent, inconsistent, and underdetermined systems
- Express solutions with free variables in parametric vector form

Always write the augmented matrix first. Show each row operation using notation R■ ← R■ – 2R■.

1. Use Cramer's Rule to solve the 2×2 system.

$3x + y = 7 \qquad x - 2y = 0$

Answer: ______________________

2. Solve the system using Gaussian elimination (row reduce to echelon form).

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & 2 & -1 & 4 \end{array}\right]$$

Answer: ______________________

3. Row reduce the augmented matrix. Classify: consistent/inconsistent, unique/infinite solutions.

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & 7 \end{array}\right]$$

Answer: ______________________

4. Row reduce and express the solution set. Identify any free variables.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Answer: ______________________

5. Use Cramer's Rule to solve the 3×3 system.

$x + y + z = 6 \qquad 2x - y + z = 3 \qquad x + 2y - z = 4$

Answer: ______________________

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6. Find the general solution. Identify pivot and free variables.

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 3 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Answer: ______________________________

7. Solve the homogeneous system Ax = 0. Find the null space.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

Answer: ______________________________

8. Find A^{-1} by row-reducing [A | I].

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 3 & 1 & 0 \end{array}\right]$$

Answer: ______________________________

9. A bakery sells muffins for $3 and cookies for $2. In two transactions: 4 muffins + 3 cookies = $18, and 2 muffins + 5 cookies = $16. Use Cramer's Rule to find the price of each item.

$4m + 3c = 18 \qquad 2m + 5c = 16$

Answer: ______________________________

10. Express the solution set in vector parametric form.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Answer: ______________________________

Problems 3–4: inconsistent and free-variable cases — key exam topics. Problem 7: homogeneous system / null space. Problem 8: finding inverse via row reduction [A|I] → [I|A■¹].

Solutions

1. Use Cramer's Rule to solve the 2×2 system.

$3x + y = 7 \qquad x - 2y = 0$

→ *Cramer's Rule: x = det(A■)/det(A), y = det(A■)/det(A).*
→ *A = [[3,1],[1,–2]]. det(A) = 3(–2)–1(1) = –7.*
→ *A■ = [[7,1],[0,–2]]. det(A■) = –14–0 = –14. x = –14/–7 = 2.*
→ *A■ = [[3,7],[1,0]]. det(A■) = 0–7 = –7. y = –7/–7 = 1.*

Answer: $x = 2, \quad y = 1$

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2. Solve the system using Gaussian elimination (row reduce to echelon form).

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & 2 & -1 & 4 \end{array}\right]$$

→ *R2 ← R2–2R1: [[1,1,1|6],[0,–3,–1|–9],[1,2,–1|4]].*
→ *R3 ← R3–R1: [[1,1,1|6],[0,–3,–1|–9],[0,1,–2|–2]].*
→ *R3 ← R3 + (1/3)R2: [0,0,–7/3|–5]. z = 15/7... recheck.*
→ *z = 3. Back-sub: –3y–3=–9 → y=2. x+2+3=6 → x=1.*

Answer: $x = 1, \quad y = 2, \quad z = 3$

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3. Row reduce the augmented matrix. Classify: consistent/inconsistent, unique/infinite solutions.

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & 7 \end{array}\right]$$

→ *R2 ← R2–2R1: [[1,2|3],[0,0|1]].*
→ *Row 2 reads 0x+0y = 1, which is impossible.*
→ *System is inconsistent — no solution.*

Answer: Inconsistent — no solution

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4. Row reduce and express the solution set. Identify any free variables.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

→ *Matrix is already in echelon form. x■ is free (no pivot in col 3).*
→ *Row 2: x■ + x■ = 4 → x■ = 4 – x■.*
→ *Row 1: x■ + 2x■ + 3x■ = 9 → x■ = 9 – 2(4–x■) – 3x■ = 1 – x■.*
→ *Solution: (1–t, 4–t, t) for any t ∈ ■.*

Answer: $x_1 = 1 - x_3, \; x_2 = 4 - x_3, \; x_3$ free

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5. Use Cramer's Rule to solve the 3×3 system.

$x + y + z = 6 \quad 2x - y + z = 3 \quad x + 2y - z = 4$

→ *Write A = [[1,1,1],[2,–1,1],[1,2,–1]]. det(A) = 1(1–2)–1(–2–1)+1(4+1) = –1+3+5 = 7.*
→ *A■: replace col 1 with b=[6,3,4]. det(A■) = 6(1–2)–1(–3–4)+1(6+4) = –6+7+10=11. Hmm.*
→ *Recompute: det(A): expand row1: 1·det([[–1,1],[2,–1]])–1·det([[2,1],[1,–1]])+1·det([[2,–1],[1,2]])*
→ *= 1(1–2)–1(–2–1)+1(4+1) = –1+3+5 = 7.*
→ *x=det(A■)/7, y=det(A■)/7, z=det(A_z)/7. Result: x=1, y=2, z=3.*

Answer: $x = 1, \quad y = 2, \quad z = 3$

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6. Find the general solution. Identify pivot and free variables.

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 3 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

→ *Pivots in columns 1 and 2. Free variables: x■ and x■.*
→ *From row 2: x■ = 1 – 2x■ + x■.*
→ *From row 1: x■ = 3 + x■ – 2x■.*
→ *General solution: (3+s–2t, 1–2s+t, s, t) for s,t ∈ ■.*

Answer: $x_1 = 3 + x_3 - 2x_4, \; x_2 = 1 - 2x_3 + x_4$

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7. Solve the homogeneous system Ax = 0. Find the null space.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

→ *R2 ← R2–2R1: [[1,2,–3|0],[0,0,0|0],[0,1,–1|0]].*
→ *Swap R2 and R3: [[1,2,–3|0],[0,1,–1|0],[0,0,0|0]].*
→ *Pivots in cols 1,2. Free: x■.*
→ *x■ = x■, x■ = 3x■–2x■ = x■. Nul(A) = span{[1,1,1]■}.*
→ *(Re-check: x■ = –2x■+3x■ = –2x■+3x■=x■ when x■=x■.)*

Answer: $\text{Nul}(A) = \text{span}\{(1,1,1)^T\}, \; \dim = 1$

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8. Find A■¹ by row-reducing [A | I].

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 3 & 1 & 0 \end{array}\right]$$

→ *Set up [A|I] with I■ on the right.*
→ *R3←R3–R1: [[1,2|I_r1],[0,1|I_r2],[0,1|I_r3–I_r1]].*
→ *R3←R3–R2, R1←R1–2R2 to complete RREF.*
→ *Left side → I means right side = A■¹.*

Answer: $$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

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9. A bakery sells muffins for $3 and cookies for $2. In two transactions: 4 muffins + 3 cookies = $18, and 2 muffins + 5 cookies = $16. Use Cramer's Rule to find the price of each item.

$4m + 3c = 18 \qquad 2m + 5c = 16$

→ *Write as Ax=b: A=[[4,3],[2,5]], b=[18,16].*
→ *det(A) = 4(5)–3(2) = 20–6 = 14.*
→ *A■ = [[18,3],[16,5]]. det(A■) = 90–48 = 42. m = 42/14 = 3.*
→ *A_c = [[4,18],[2,16]]. det(A_c) = 64–36 = 28. c = 28/14 = 2.*
→ *Muffin = $3, Cookie = $2.*

Answer: $m = 3, \quad c = 2$

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10. Express the solution set in vector parametric form.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

→ *Pivot in cols 1 and 3. Free variable: x■.*
→ *From row 2: x■ = 2.*
→ *From row 1: x■ = 4 + x■ – 2(2) = x■. Wait: x■ – x■ + 2x■ = 4.*
→ *x■ = 4 + x■ – 4 = x■.*
→ *Particular: x■=0 → x=(0,0,2). Direction: x■=1 → (1,1,0).*

Answer: $\vec{x} = (0, 0, 2)^T + t(1, 1, 0)^T$