MATH 250 — Vector Spaces

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Learning Objectives

- Test sets of vectors for linear independence
- Determine if a set of vectors spans a given space
- Find a basis and dimension for null space and column space
- Apply the Rank-Nullity Theorem and Gram-Schmidt process

For subspace proofs, check all three axioms: zero vector, closure under addition, closure under scalar multiplication.

1. Are u and v linearly independent? Set up cu + dv = 0 and solve.

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Answer: ________________________________

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2. Determine if the three vectors are linearly independent.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Answer: ________________________________

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3. Do u and v span R^2? Explain using the determinant.

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Answer: ________________________________

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4. Is W = {(x, y, z) : x + y + z = 0} a subspace of ■^{3}?

$$W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$$

Answer: ________________________________

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5. Find a basis and the dimension of the null space of A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Answer: ______________________________

6. Find rank(A) and nullity(A). Verify the Rank-Nullity Theorem.

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer: ______________________________

7. Find a basis for the column space (range) of A.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}$$

Answer: ______________________________

8. Use the Gram-Schmidt process to find an orthogonal basis for the span of v■ and v■.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Answer: ______________________________

9. Find the coordinate vector [x]_B of x with respect to basis B.

$$\vec{x} = (5, 3)^T, \quad B = \{(2, 1)^T, \ (1, 1)^T\}$$

Answer: ______________________________

10. A linear transformation T: ■² → ■² is defined by T(x) = Ax. Find T([2, –1]■).

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Answer: ______________________________

Problems 4–5: subspace check and null space basis. Problem 6 (Rank-Nullity): critical theorem, always verify n = rank + nullity. Problem 8 (Gram-Schmidt): show the projection formula explicitly.

Solutions

1. Are u and v linearly independent? Set up cu + dv = 0 and solve.

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

→ *c[1,2]■ + d[3,6]■ = 0 → c+3d=0, 2c+6d=0.*
→ *Both equations: c = –3d. Non-trivial solution exists (d=1, c=–3).*
→ *v = 3u — the vectors are parallel. Linearly dependent.*

Answer: Linearly dependent: $\vec{v} = 3\vec{u}$

2. Determine if the three vectors are linearly independent.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

→ *Form matrix A = [v■|v■|v■] and compute det(A).*
→ *det([[1,2,1],[0,1,–1],[1,3,0]]) = 1(0+3)–2(0+1)+1(0–1) = 3–2–1 = 0.*
→ *det=0 → linearly dependent.*
→ *Correction: 1(1·0–(–1)·3)–2(0·0–(–1)·1)+1(0·3–1·1)=1(3)–2(1)+1(–1)=0.*
→ *Result: det = 0 → vectors are linearly DEPENDENT.*

Answer: Linearly independent

3. Do u and v span R²? Explain using the determinant.

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

→ *det([[1,–1],[2,3]]) = 3+2 = 5 ≠ 0.*
→ *Non-zero det means the columns are linearly independent.*
→ *{u, v} is a basis for ■². ✓*

Answer: $\det = 5 \neq 0 \Rightarrow \{\vec{u}, \vec{v}\}$ spans $\mathbb{R}^2$

4. Is W = {(x, y, z) : x + y + z = 0} a subspace of ■³?

$$W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$$

→ *1) Zero vector: 0+0+0=0 ✓.*
→ *2) Closure under addition: if x■+y■+z■=0 and x■+y■+z■=0, then (x■+x■)+(y■+y■)+(z■+z■)=0 ✓.*
→ *3) Closure under scalar mult: c(x)+c(y)+c(z)=c(x+y+z)=0 ✓.*
→ *W satisfies all three subspace axioms.*

Answer: Yes, W is a subspace of $\mathbb{R}^3$

5. Find a basis and the dimension of the null space of A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

→ *Row reduce: already in echelon form. Pivots in cols 1 and 3.*
→ *Free variable: x■.*
→ *From row 2: x■=0. From row 1: x■+2x■=0 → x■=–2x■.*
→ *Null space = {(–2t, t, 0) : t ∈ ■}. Basis: {[–2,1,0]■}. dim=1.*

Answer: $\mathrm{Nul}(A) = \mathrm{span}\{(-2, 1, 0)^T\},\ \dim = 1$

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6. Find rank(A) and nullity(A). Verify the Rank-Nullity Theorem.

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

→ *A is 3×4 (n=4 columns). Already in echelon form.*
→ *Pivot columns: 1 and 3. Rank = 2.*
→ *Nullity = n – rank = 4 – 2 = 2.*
→ *Rank-Nullity: 2+2=4 ✓.*

Answer: $\mathrm{rank} = 2,\ \mathrm{nullity} = 2,\ \mathrm{rank} + \mathrm{nullity} = 4 = n$

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7. Find a basis for the column space (range) of A.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}$$

→ *Row reduce A to find pivot columns.*
→ *R2←R2–2R1, R3←R3–3R1: [[1,2,1],[0,0,1],[0,0,1]].*
→ *R3←R3–R2: [[1,2,1],[0,0,1],[0,0,0]]. Pivots in cols 1 and 3.*
→ *Basis for Col(A) = original pivot columns = [1,2,3]■ and [1,3,4]■.*

Answer: $\mathrm{Col}(A) = \mathrm{span}\{(1, 2, 3)^T,\ (1, 3, 4)^T\}$

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8. Use the Gram-Schmidt process to find an orthogonal basis for the span of v■ and v■.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

→ *u■ = v■ = [1,1,0]■.*
→ *proj_{u■}(v■) = (v■·u■/u■·u■)u■ = (1/2)[1,1,0]■ = [1/2,1/2,0]■.*
→ *u■ = v■ – proj = [1,0,1]■ – [1/2,1/2,0]■ = [1/2,–1/2,1]■.*
→ *Verify: u■·u■ = 1/2–1/2+0 = 0 ✓.*

Answer: $\vec{u}_1 = (1, 1, 0)^T, \quad \vec{u}_2 = (1/2, -1/2, 1)^T$

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9. Find the coordinate vector [x]_B of x with respect to basis B.

$\vec{x} = (5, 3)^T, \quad B = \{(2, 1)^T, (1, 1)^T\}$

→ Solve c■[2,1]■ + c■[1,1]■ = [5,3]■.
→ 2c■+c■=5 and c■+c■=3.
→ Subtract: c■=2. Then c■=1.
→ [x]_B = [2,1]■.

Answer: $[\vec{x}]_B = (2, 1)^T$

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10. A linear transformation T: ■² → ■² is defined by T(x) = Ax. Find T([2, –1]■).

$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

→ T(x) = Ax = [[1,2],[3,–1]]·[2,–1]■.
→ Row 1: 1(2)+2(–1) = 0.
→ Row 2: 3(2)+(–1)(–1) = 7.
→ T([2,–1]■) = [0,7]■.

Answer: $\begin{bmatrix} 0 \\ 7 \end{bmatrix}$