



Optimization: Maximum Volume of an Open-Top Box

Calculus Worksheet · Grade 11-12

Name: _____

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Learning Objectives

- Model the volume of an open-top box as a function of the cut-out side length
- Determine the valid domain restrictions for a real-world optimization problem
- Apply derivatives to find the dimensions that maximize volume

For each problem, set up the volume function, state the domain, and use calculus to find the maximum volume where required.

1. A square of side x inches is cut from each corner of an 8 by 15 inch cardboard. Write the volume V of the resulting open-top box as a function of x in factored form.

$$V(x) = \text{length} \cdot \text{width} \cdot \text{height}$$

Answer: _____

2. Expand the volume function $V(x) = x(8 - 2x)(15 - 2x)$ into standard polynomial form.

$$V(x) = x(8 - 2x)(15 - 2x)$$

Answer: _____

3. State the restriction on the domain of $V(x)$ so the box has physically meaningful dimensions.

$$V(x) = x(8 - 2x)(15 - 2x)$$

Answer: _____

4. Find the derivative $V'(x)$ for $V(x) = 4x^3 - 46x^2 + 120x$.

$$V(x) = 4x^3 - 46x^2 + 120x$$

Answer: _____

5. Find the critical numbers of $V(x)$ by solving $V'(x) = 0$.

$$12x^2 - 92x + 120 = 0$$

Answer: _____

6. Identify which critical number lies in the domain $0 < x < 4$ and represents a candidate for the maximum.

$$x = \frac{5}{3}, x = 6$$

Answer: _____



7. Use the second derivative test to confirm that $x = 5/3$ yields a maximum volume.

$$V''(x) = 24x - 92$$

Answer: _____

8. Find the dimensions of the box (length, width, height) that produce the maximum volume.

$$x = \frac{5}{3}$$

Answer: _____

9. Compute the maximum volume of the box (in cubic inches) by evaluating $V(5/3)$.

$$V\left(\frac{5}{3}\right) = \frac{5}{3}\left(8 - \frac{10}{3}\right)\left(15 - \frac{10}{3}\right)$$

Answer: _____

10. A square of side x is cut from each corner of a 10 by 20 inch cardboard to form an open-top box. Write $V(x)$ and state the domain.

$$V(x) = x(10 - 2x)(20 - 2x)$$

Answer: _____





Emphasize that students must verify critical points lie within the domain and use the first or second derivative test to confirm a maximum.

Solutions

1. A square of side x inches is cut from each corner of an 8 by 15 inch cardboard. Write the volume V of the resulting open-top box as a function of x in factored form.

$$V(x) = \text{length} \cdot \text{width} \cdot \text{height}$$

- The original cardboard is 8 inches by 15 inches.
- After cutting squares of side x from each corner and folding up, the height becomes x .
- The width becomes 8 minus $2x$ because x is removed from both sides.
- The length becomes 15 minus $2x$ because x is removed from both ends.
- Multiply length times width times height to get V of x equals x times (8 minus $2x$) times (15 minus $2x$).

Answer: $V(x) = x(8 - 2x)(15 - 2x)$

2. Expand the volume function $V(x) = x(8 - 2x)(15 - 2x)$ into standard polynomial form.

$$V(x) = x(8 - 2x)(15 - 2x)$$

- First multiply (8 minus $2x$) by (15 minus $2x$) to get 120 minus $16x$ minus $30x$ plus $4x$ squared.
- Combine like terms to get $4x$ squared minus $46x$ plus 120.
- Multiply the result by x to distribute.
- The expanded form is $4x$ cubed minus $46x$ squared plus $120x$.

Answer: $V(x) = 4x^3 - 46x^2 + 120x$

3. State the restriction on the domain of $V(x)$ so the box has physically meaningful dimensions.

$$V(x) = x(8 - 2x)(15 - 2x)$$

- The height x must be positive, so x is greater than 0.
- The width 8 minus $2x$ must be positive, giving x less than 4.
- The length 15 minus $2x$ must be positive, giving x less than 7.5.
- The most restrictive bound is x less than 4, so the domain is 0 less than x less than 4.

Answer: $0 < x < 4$

4. Find the derivative $V'(x)$ for $V(x) = 4x^3 - 46x^2 + 120x$.

$$V(x) = 4x^3 - 46x^2 + 120x$$

- Apply the power rule to each term of V of x .
- The derivative of $4x$ cubed is $12x$ squared.
- The derivative of negative $46x$ squared is negative $92x$.
- The derivative of $120x$ is 120.
- Combine to get V prime of x equals $12x$ squared minus $92x$ plus 120.

Answer: $V'(x) = 12x^2 - 92x + 120$



5. Find the critical numbers of $V(x)$ by solving $V'(x) = 0$.

$$12x^2 - 92x + 120 = 0$$

- Set $12x$ squared minus $92x$ plus 120 equal to 0 .
- Divide both sides by 4 to simplify to $3x$ squared minus $23x$ plus 30 equals 0 .
- Use the quadratic formula or factoring to obtain $(3x - 5)(x - 6)$ equals 0 .
- Solving gives x equals 5 over 3 or x equals 6 .

Answer: $x = \frac{5}{3}, x = 6$

6. Identify which critical number lies in the domain $0 < x < 4$ and represents a candidate for the maximum.

$$x = \frac{5}{3}, x = 6$$

- Compare each critical number to the domain 0 less than x less than 4 .
- The value 6 is outside the domain because it is greater than 4 .
- The value 5 over 3 is approximately 1.67 , which lies inside the domain.
- Therefore the only candidate inside the domain is x equals 5 over 3 .

Answer: $x = \frac{5}{3}$

7. Use the second derivative test to confirm that $x = 5/3$ yields a maximum volume.

$$V''(x) = 24x - 92$$

- Differentiate V prime of x equals $12x$ squared minus $92x$ plus 120 to get V double prime of x equals $24x$ minus 92 .
- Substitute x equals 5 over 3 into V double prime of x .
- Compute 24 times 5 over 3 equals 40 , then 40 minus 92 equals negative 52 .
- Since V double prime is negative at x equals 5 over 3 , the function is concave down and the point is a local maximum.

Answer: $V''\left(\frac{5}{3}\right) = -52 < 0 \Rightarrow \text{maximum}$

8. Find the dimensions of the box (length, width, height) that produce the maximum volume.

$$x = \frac{5}{3}$$

- The height equals x , which is 5 over 3 inches.
- The width equals 8 minus $2x$, which is 8 minus 10 over 3 equals 14 over 3 inches.
- The length equals 15 minus $2x$, which is 15 minus 10 over 3 equals 35 over 3 inches.
- So the box measures 35 over 3 by 14 over 3 by 5 over 3 inches.

Answer: Height = $\frac{5}{3}$, Width = $\frac{14}{3}$, Length = $\frac{35}{3}$

9. Compute the maximum volume of the box (in cubic inches) by evaluating $V(5/3)$.

$$V\left(\frac{5}{3}\right) = \frac{5}{3}\left(8 - \frac{10}{3}\right)\left(15 - \frac{10}{3}\right)$$

- Substitute x equals 5 over 3 into V of x equals x times $(8$ minus $2x)$ times $(15$ minus $2x)$.
- Compute 8 minus 10 over 3 equals 14 over 3 and 15 minus 10 over 3 equals 35 over 3 .
- Multiply 5 over 3 times 14 over 3 times 35 over 3 to get 2450 over 27 .
- This value is approximately 90.74 cubic inches.

Answer: $V_{\max} = \frac{2450}{27} \approx 90.74 \text{ in}^3$



10. A square of side x is cut from each corner of a 10 by 20 inch cardboard to form an open-top box. Write $V(x)$ and state the domain.

$$V(x) = x(10 - 2x)(20 - 2x)$$

→ The height equals x and must be positive.

→ The width 10 minus $2x$ must be positive so x is less than 5.

→ The length 20 minus $2x$ must be positive so x is less than 10.

→ The binding restriction is x less than 5, giving the domain 0 less than x less than 5.

→ Expanding gives V of x equals $4x$ cubed minus $60x$ squared plus $200x$.

Answer: $V(x) = 4x^3 - 60x^2 + 200x, 0 < x < 5$

