Optimization: Open-Top Box Problem

Calculus Worksheet · Grade 11–12

Name: ________________________________ Date: ___________________

Learning Objectives

- Express the volume of an open-top box as a function of the corner cut size x
- Determine the domain restrictions for the volume function based on physical constraints
- Use derivatives to find the value of x that maximizes the volume of the box

Problems

1. A square of side x inches is cut from each corner of a 10 by 10 inch cardboard. The sides are folded up to form an open-top box. Write an expression for the height of the box in terms of x.

$$h = x$$

2. A square of side x inches is cut from each corner of an 8 by 15 inch cardboard and the sides are folded up. Write an expression for the length and the width of the resulting open-top box in terms of x.

$$\text{Length} = 8 - 2x, \quad \text{Width} = 15 - 2x$$

3. Using the 8 by 15 inch cardboard with corner squares of side x cut out, write the volume V of the open-top box as a function of x.

$$V(x) = x(8 - 2x)(15 - 2x)$$

4. Expand and simplify the volume function for the 8 by 15 inch open-top box, V(x) = x(8 – 2x)(15 – 2x), into standard polynomial form.

$$V(x) = x(8 - 2x)(15 - 2x)$$

5. For the 8 by 15 inch open-top box, find the domain restriction on x by setting each dimension (length, width, and height) greater than or equal to zero. Write the final restricted domain as an inequality.

$$8 - 2x \geq 0, \quad 15 - 2x \geq 0, \quad x \geq 0$$

Scan to watch

6. A square of side x is cut from each corner of a 6 by 12 inch cardboard to form an open-top box. Write the volume as a function of x in standard polynomial form, and state the domain restriction.

$$V(x) = x(6 - 2x)(12 - 2x)$$

7. Find the derivative of the volume function $V(x) = 4x^3 - 46x^2 + 120x$ for the 8 by 15 inch open-top box.

$$V(x) = 4x^3 - 46x^2 + 120x$$

8. Using the derivative $V'(x) = 12x^2 - 92x + 120$, set $V'(x) = 0$ and solve for x to find the critical points. Then determine which critical point lies within the valid domain $0 < x < 4$.

$$12x^2 - 92x + 120 = 0$$

9. Using the second derivative test, confirm whether $x = 5/3$ gives a maximum volume for the 8 by 15 inch open-top box. Find the second derivative $V''(x)$ and evaluate it at $x = 5/3$.

$$V'(x) = 12x^2 - 92x + 120$$

10. Find the maximum volume of the open-top box formed from the 8 by 15 inch cardboard by substituting $x = 5/3$ into $V(x) = x(8 - 2x)(15 - 2x)$. Round your answer to two decimal places.

$$V\left(\tfrac{5}{3}\right) = \tfrac{5}{3}\left(8 - \tfrac{10}{3}\right)\left(15 - \tfrac{10}{3}\right)$$

Scan to watch

Optimization: Open-Top Box Problem — Answer Key

Calculus Worksheet · Grade 11–12

Answer Key

1. Answer: $h = x$

- When the corner squares of side x are folded up, they become the walls of the box.
- Therefore the height of the open-top box equals the side length of the cut-out square: $h = x$.

2. Answer: Length = $8 - 2x$, Width = $15 - 2x$

- Two corners are removed from the 8-inch side, each of size x, so the new length = $8 - 2x$.
- Two corners are removed from the 15-inch side, each of size x, so the new width = $15 - 2x$.

3. Answer: $V(x) = x(8 - 2x)(15 - 2x)$

- Volume of a rectangular box = length × width × height.
- Length = $8 - 2x$, Width = $15 - 2x$, Height = x.
- Therefore $V(x) = x(8 - 2x)(15 - 2x)$.

4. Answer: $V(x) = 4x^3 - 46x^2 + 120x$

- First expand $(8 - 2x)(15 - 2x) = 120 - 16x - 30x + 4x^2 = 120 - 46x + 4x^2$.
- Now multiply by x: $V(x) = 120x - 46x^2 + 4x^3$.
- Written in standard form: $V(x) = 4x^3 - 46x^2 + 120x$.

5. Answer: $0 < x < 4$

- From $8 - 2x \geq 0$: $x \leq 4$.
- From $15 - 2x \geq 0$: $x \leq 7.5$.
- From height: $x \geq 0$.
- The most restrictive upper bound is $x \leq 4$, and x must be strictly positive for a real box.
- Therefore the restricted domain is $0 < x < 4$.

6. Answer: $V(x) = 4x^3 - 36x^2 + 72x$, domain: $0 < x < 3$

- Length = $6 - 2x$, Width = $12 - 2x$, Height = x.
- Expand $(6 - 2x)(12 - 2x) = 72 - 12x - 24x + 4x^2 = 72 - 36x + 4x^2$.
- Multiply by x: $V(x) = 72x - 36x^2 + 4x^3 = 4x^3 - 36x^2 + 72x$.
- Domain: $6 - 2x > 0 \rightarrow x < 3$; $12 - 2x > 0 \rightarrow x < 6$; $x > 0$.
- Most restrictive: $0 < x < 3$.

7. Answer: $V'(x) = 12x^2 - 92x + 120$

- Differentiate each term using the power rule.
- $\frac{d}{dx}(4x^3) = 12x^2$, $\frac{d}{dx}(-46x^2) = -92x$, $\frac{d}{dx}(120x) = 120$.
- Therefore $V'(x) = 12x^2 - 92x + 120$.

Scan to watch

8. Answer: $x \approx 1.87$ (the critical point inside $0 < x < 4$)

- Divide both sides by 4: $3x^2 - 23x + 30 = 0$.
- Use the quadratic formula: $x = (23 \pm \sqrt{(529 - 360)}) / 6 = (23 \pm \sqrt{169}) / 6 = (23 \pm 13) / 6$.
- $x = 36/6 = 6$ or $x = 10/6 \approx 1.67$.
- Check domain $0 < x < 4$: $x = 6$ is outside; $x \approx 1.67$ is inside.
- The critical point in the valid domain is $x = 5/3 \approx 1.67$.

9. Answer: $V''(5/3) = -52 < 0$, confirming a local maximum

- Differentiate $V'(x) = 12x^2 - 92x + 120$ to get $V''(x) = 24x - 92$.
- Evaluate at $x = 5/3$: $V''(5/3) = 24(5/3) - 92 = 40 - 92 = -52$.
- Since $V''(5/3) = -52 < 0$, the function is concave down at this point.
- Therefore $x = 5/3$ gives a local (and global, within the domain) maximum volume.

10. Answer: $V_{max} \approx 90.74$ cubic inches

- Substitute $x = 5/3$ into each dimension.
- Height = 5/3; Length = 8 – 10/3 = 14/3; Width = 15 – 10/3 = 35/3.
- $V = (5/3)(14/3)(35/3) = (5 \times 14 \times 35) / 27 = 2450 / 27 \approx 90.74$.
- The maximum volume of the open-top box is approximately 90.74 cubic inches.

Scan to watch