

# Optimization Problems Using Derivatives

Calculus Worksheet · Grade 11–12 / College Calculus

Name: \_\_\_\_\_

Date: \_\_\_\_\_

## Learning Objectives

- Set up two equations (constraint and objective) from a word problem
- Use substitution to reduce the objective function to a single variable and apply differentiation
- Verify maximum or minimum values using the first derivative sign chart

## Problems

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1. Find two positive numbers whose sum is 60 such that their product is maximized.

$$x + y = 60, \quad P = xy$$

2. Find two positive numbers such that the sum of the first and three times the second is 120, and their product is maximized.

$$x + 3y = 120, \quad P = xy$$

3. A farmer has 200 meters of fencing and wants to enclose a rectangular garden. One side of the garden borders a barn wall (no fencing needed on that side). Find the dimensions that maximize the enclosed area.

$$2W + L = 200, \quad A = LW$$

4. Find two positive numbers such that their sum is 80 and the sum of their squares is minimized.

$$x + y = 80, \quad S = x^2 + y^2$$

5. A rectangular garden is to be enclosed using 300 meters of fencing for all four sides. Find the dimensions that maximize the area.

$$2L + 2W = 300, \quad A = LW$$

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6. Find a positive number such that the sum of the number and its reciprocal is minimized.

$$f(x) = x + \frac{1}{x}, \quad x > 0$$

7. A box with a square base and no top is to be made from a sheet of metal with a total surface area of 108 square centimeters. Find the dimensions of the box that maximize its volume.

$$x^2 + 4xh = 108, \quad V = x^2h$$

8. A Norman window consists of a rectangle topped with a semicircle. If the perimeter of the window is 10 meters, find the radius of the semicircle and the height of the rectangular portion that maximize the area.

$$2h + 2r + \pi r = 10, \quad A = 2rh + \frac{\pi r^2}{2}$$

9. A cylindrical can is to hold 500 cubic centimeters of liquid. Find the radius and height of the can that minimize the total surface area (including both circular ends).

$$\pi r^2 h = 500, \quad S = 2\pi r^2 + 2\pi r h$$

10. A company sells  $x$  units of a product per week. The weekly revenue function is  $R(x) = 200x - 0.5x^2$  and the weekly cost function is  $C(x) = 50x + 2000$ . Find the number of units that maximizes the weekly profit, and state the maximum profit.

$$P(x) = R(x) - C(x) = 150x - 0.5x^2 - 2000$$

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# Optimization Problems Using Derivatives — Answer Key

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## Answer Key

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### 1. Answer: $x = 30$ , $y = 30$ ; Maximum product = 900

- Constraint:  $x + y = 60$ , so  $x = 60 - y$
- Objective:  $P = xy = (60 - y)y = 60y - y^2$
- Derivative:  $P'(y) = 60 - 2y$
- Set  $P'(y) = 0$ :  $60 - 2y = 0 \rightarrow y = 30$
- Sign chart:  $P'(29) > 0$  and  $P'(31) < 0$ , confirming a maximum at  $y = 30$
- $x = 60 - 30 = 30$ ; Maximum product =  $30 \times 30 = 900$

### 2. Answer: $x = 60$ , $y = 20$ ; Maximum product = 1200

- Constraint:  $x + 3y = 120$ , so  $x = 120 - 3y$
- Objective:  $P = xy = (120 - 3y)y = 120y - 3y^2$
- Derivative:  $P'(y) = 120 - 6y$
- Set  $P'(y) = 0$ :  $120 - 6y = 0 \rightarrow y = 20$
- Sign chart:  $P'(19) > 0$  and  $P'(21) < 0$ , confirming maximum at  $y = 20$
- $x = 120 - 3(20) = 60$ ; Maximum product =  $60 \times 20 = 1200$

### 3. Answer: $W = 50$ m, $L = 100$ m; Maximum area = 5000 m<sup>2</sup>

- Constraint:  $2W + L = 200$ , so  $L = 200 - 2W$
- Objective:  $A = LW = (200 - 2W)W = 200W - 2W^2$
- Derivative:  $A'(W) = 200 - 4W$
- Set  $A'(W) = 0$ :  $200 - 4W = 0 \rightarrow W = 50$
- Sign chart:  $A'(49) > 0$  and  $A'(51) < 0$ , confirming maximum at  $W = 50$
- $L = 200 - 2(50) = 100$ ; Maximum area =  $100 \times 50 = 5000$  m<sup>2</sup>

### 4. Answer: $x = 40$ , $y = 40$ ; Minimum sum of squares = 3200

- Constraint:  $x + y = 80$ , so  $x = 80 - y$
- Objective:  $S = x^2 + y^2 = (80 - y)^2 + y^2 = 6400 - 160y + 2y^2$
- Derivative:  $S'(y) = -160 + 4y$
- Set  $S'(y) = 0$ :  $-160 + 4y = 0 \rightarrow y = 40$
- Sign chart:  $S'(39) < 0$  and  $S'(41) > 0$ , confirming minimum at  $y = 40$
- $x = 80 - 40 = 40$ ; Minimum  $S = 40^2 + 40^2 = 3200$

### 5. Answer: $L = 75$ m, $W = 75$ m; Maximum area = 5625 m<sup>2</sup>

- Constraint:  $2L + 2W = 300 \rightarrow L + W = 150$ , so  $L = 150 - W$
- Objective:  $A = LW = (150 - W)W = 150W - W^2$
- Derivative:  $A'(W) = 150 - 2W$
- Set  $A'(W) = 0$ :  $150 - 2W = 0 \rightarrow W = 75$

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- Sign chart:  $A'(74) > 0$  and  $A'(76) < 0$ , confirming maximum at  $W = 75$
- $L = 150 - 75 = 75$ ; Maximum area =  $75 \times 75 = 5625 \text{ m}^2$

**6. Answer:  $x = 1$ ; Minimum value = 2**

- Objective function:  $f(x) = x + 1/x$
- Derivative:  $f'(x) = 1 - 1/x^2$
- Set  $f'(x) = 0$ :  $1 - 1/x^2 = 0 \rightarrow x^2 = 1 \rightarrow x = 1$  (positive)
- Sign chart:  $f'(0.5) = 1 - 4 = -3 < 0$  and  $f'(2) = 1 - 1/4 = 0.75 > 0$ , confirming minimum at  $x = 1$
- Minimum value:  $f(1) = 1 + 1/1 = 2$

**7. Answer:  $x = 6 \text{ cm}$ ,  $h = 3 \text{ cm}$ ; Maximum volume =  $108 \text{ cm}^3$**

- Constraint:  $x^2 + 4xh = 108$ , so  $h = (108 - x^2) / (4x)$
- Objective:  $V = x^2h = x^2 \cdot (108 - x^2)/(4x) = (108x - x^3)/4$
- Derivative:  $V'(x) = (108 - 3x^2)/4$
- Set  $V'(x) = 0$ :  $108 - 3x^2 = 0 \rightarrow x^2 = 36 \rightarrow x = 6$
- Sign chart:  $V'(5) > 0$  and  $V'(7) < 0$ , confirming maximum at  $x = 6$
- $h = (108 - 36)/(4 \cdot 6) = 72/24 = 3$ ;  $V = 36 \times 3 = 108 \text{ cm}^3$

**8. Answer:  $r = 10 / (4 + \pi) \approx 1.40 \text{ m}$ ;  $h = r/2 \cdot \pi \approx 1.40 \text{ m}$**

- Constraint:  $2h + 2r + \pi r = 10$ , so  $h = (10 - 2r - \pi r)/2 = 5 - r - (\pi/2)r$
- Objective:  $A = 2rh + (\pi r^2)/2$ ; substitute  $h$ :  $A = 2r(5 - r - \pi r/2) + \pi r^2/2$
- Simplify:  $A = 10r - 2r^2 - \pi r^2 + \pi r^2/2 = 10r - 2r^2 - (\pi/2)r^2$
- Derivative:  $A'(r) = 10 - 4r - \pi r$
- Set  $A'(r) = 0$ :  $10 = r(4 + \pi) \rightarrow r = 10/(4 + \pi) \approx 1.40 \text{ m}$
- $h = 5 - r - (\pi/2)r = 5 - r(1 + \pi/2) \approx 1.40 \text{ m}$

**9. Answer:  $r = \sqrt[3]{(250/\pi)} \approx 4.30 \text{ cm}$ ,  $h \approx 8.60 \text{ cm}$ ; Minimum surface area  $\approx 348.7 \text{ cm}^2$**

- Constraint:  $\pi r^2 h = 500$ , so  $h = 500/(\pi r^2)$
- Objective:  $S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \cdot 500/(\pi r^2) = 2\pi r^2 + 1000/r$
- Derivative:  $S'(r) = 4\pi r - 1000/r^2$
- Set  $S'(r) = 0$ :  $4\pi r = 1000/r^2 \rightarrow r^3 = 1000/(4\pi) = 250/\pi \rightarrow r = \sqrt[3]{(250/\pi)} \approx 4.30 \text{ cm}$
- Sign chart:  $S'(4) < 0$  and  $S'(5) > 0$ , confirming minimum at  $r \approx 4.30$
- $h = 500/(\pi \cdot (250/\pi)^{(2/3)}) = 2r \approx 8.60 \text{ cm}$

**10. Answer:  $x = 150$  units; Maximum profit = \$9,250**

- Profit:  $P(x) = R(x) - C(x) = (200x - 0.5x^2) - (50x + 2000) = 150x - 0.5x^2 - 2000$
- Derivative:  $P'(x) = 150 - x$
- Set  $P'(x) = 0$ :  $150 - x = 0 \rightarrow x = 150$
- Sign chart:  $P'(149) = 1 > 0$  and  $P'(151) = -1 < 0$ , confirming maximum at  $x = 150$
- Maximum profit:  $P(150) = 150(150) - 0.5(150^2) - 2000 = 22500 - 11250 - 2000 = 9250$
- The company should sell 150 units per week for a maximum weekly profit of \$9,250

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