Optimization Word Problems Using Derivatives

Calculus Worksheet · Grade 11–12 / College Calculus

Name: ________________________________ Date: __________________

Learning Objectives

- Set up equations from word problems using one variable to model optimization scenarios
- Find critical numbers by taking the derivative of an objective function and setting it equal to zero
- Use a sign chart (first derivative test) to confirm whether a critical point is a minimum or maximum

Problems

1. Find two numbers whose difference is 20 and whose product is minimized.

$$x - y = 20$$

2. Find two positive numbers whose sum is 50 and whose product is maximized.

$$x + y = 50$$

3. A farmer has 200 feet of fencing and wants to enclose a rectangular garden. Find the dimensions that maximize the area.

$$P = 2l + 2w = 200$$

4. Find two positive numbers whose sum is 100 and the sum of whose squares is minimized.

$$x + y = 100$$

5. A rectangle has a perimeter of 120 meters. A fence is built on three sides (no fence on one long side because it borders a wall). Find the dimensions that maximize the enclosed area.

$$l + 2w = 120$$

6. Find the dimensions of a rectangle with area 400 square centimeters such that the perimeter is minimized.

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$$lw = 400$$

7. A box with a square base and no top must have a volume of 32 cubic feet. Find the dimensions that minimize the surface area.

$$V = x^2h = 32$$

8. A company manufactures and sells x units of a product per week. The cost function is given below and the revenue function is also given below. Find the production level x that maximizes profit, where profit equals revenue minus cost.

$$\begin{cases} C(x) = 500 + 20x + 0.5x^2 \\ R(x) = 80x - x^2 \\ P(x) = R(x) - C(x) \end{cases}$$

9. A cylindrical can (with top and bottom) must hold exactly 54 pi cubic inches. Find the radius and height that minimize the total surface area of the can.

$$V = \pi r^2 h = 54\pi$$

10. A Norman window consists of a rectangle topped by a semicircle. The total perimeter of the window (the outer edge) is 12 meters. Find the radius of the semicircle and the height of the rectangular portion that maximize the total area of the window.

$$2h + 2r + \pi r = 12$$

Optimization Word Problems Using Derivatives — Answer Key

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Answer Key

1. Answer: $x = -10$ and $y = -10$; minimum product = 100

- Let x and y be the two numbers. Constraint: $x - y = 20$, so $x = y + 20$.
- Product function: $P = xy = (y + 20)y = y^2 + 20y$.
- Differentiate: $P'(y) = 2y + 20$. Set $P'(y) = 0 \rightarrow y = -10$.
- Then $x = -10 + 20 = 10$.
- Sign chart: $P'(y) < 0$ for $y < -10$ and $P'(y) > 0$ for $y > -10 \rightarrow$ minimum at $y = -10$.
- The two numbers are 10 and -10, with minimum product = $(10)(-10) = -100$.

2. Answer: $x = 25$ and $y = 25$; maximum product = 625

- Let x and y be the two positive numbers with $x + y = 50$, so $y = 50 - x$.
- Product: $P = x(50 - x) = 50x - x^2$.
- Differentiate: $P'(x) = 50 - 2x$. Set $P'(x) = 0 \rightarrow x = 25$.
- Then $y = 50 - 25 = 25$.
- Sign chart: $P'(x) > 0$ for $x < 25$ and $P'(x) < 0$ for $x > 25 \rightarrow$ maximum at $x = 25$.
- Maximum product = $25 \times 25 = 625$.

3. Answer: $l = 50$ ft and $w = 50$ ft; maximum area = 2500 sq ft

- Perimeter constraint: $2l + 2w = 200$, so $l = 100 - w$.
- Area function: $A = lw = (100 - w)w = 100w - w^2$.
- Differentiate: $A'(w) = 100 - 2w$. Set $A'(w) = 0 \rightarrow w = 50$.
- Then $l = 100 - 50 = 50$.
- Sign chart confirms maximum at $w = 50$.
- Maximum area = $50 \times 50 = 2500$ sq ft. The garden should be a 50 ft × 50 ft square.

4. Answer: $x = 50$ and $y = 50$; minimum sum of squares = 5000

- Constraint: $x + y = 100$, so $y = 100 - x$.
- Objective: $S = x^2 + y^2 = x^2 + (100 - x)^2 = 2x^2 - 200x + 10000$.
- Differentiate: $S'(x) = 4x - 200$. Set $S'(x) = 0 \rightarrow x = 50$.
- Then $y = 100 - 50 = 50$.
- Sign chart: $S'(x) < 0$ for $x < 50$ and $S'(x) > 0$ for $x > 50 \rightarrow$ minimum at $x = 50$.
- Minimum sum of squares = $50^2 + 50^2 = 5000$.

5. Answer: $w = 30$ m and $l = 60$ m; maximum area = 1800 sq m

- Fencing constraint (3 sides): $l + 2w = 120$, so $l = 120 - 2w$.
- Area: $A = lw = (120 - 2w)w = 120w - 2w^2$.
- Differentiate: $A'(w) = 120 - 4w$. Set $A'(w) = 0 \rightarrow w = 30$.
- Then $l = 120 - 2(30) = 60$.

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- Sign chart: $A'(w) > 0$ for $w < 30$ and $A'(w) < 0$ for $w > 30$ → maximum at $w = 30$.
- Maximum area = $60 \times 30 = 1800$ sq m.

6. Answer: l = 20 cm and w = 20 cm; minimum perimeter = 80 cm

- Constraint: $lw = 400$, so $l = 400/w$.
- Perimeter: $P = 2l + 2w = 2(400/w) + 2w = 800/w + 2w$.
- Differentiate: $P'(w) = -800/w^2 + 2$. Set $P'(w) = 0 \rightarrow 2 = 800/w^2 \rightarrow w^2 = 400 \rightarrow w = 20$.
- Then $l = 400/20 = 20$.
- Sign chart confirms minimum at $w = 20$ ($P'' > 0$ confirming concave up).
- Minimum perimeter = $2(20) + 2(20) = 80$ cm.

7. Answer: x = 4 ft and h = 2 ft; minimum surface area = 48 sq ft

- Let x = side of square base, h = height. Volume: $x^2h = 32$, so $h = 32/x^2$.
- Surface area (no top): $S = x^2 + 4xh = x^2 + 4x(32/x^2) = x^2 + 128/x$.
- Differentiate: $S'(x) = 2x - 128/x^2$. Set $S'(x) = 0 \rightarrow 2x = 128/x^2 \rightarrow x^3 = 64 \rightarrow x = 4$.
- Then $h = 32/16 = 2$.
- $S''(x) = 2 + 256/x^3 > 0$ → confirms minimum.
- Minimum surface area = $4^2 + 4(4)(2) = 16 + 32 = 48$ sq ft.

8. Answer: x = 20 units; maximum profit = P(20) = 100

- Profit: $P(x) = R(x) - C(x) = (80x - x^2) - (500 + 20x + 0.5x^2) = 60x - 1.5x^2 - 500$.
- Differentiate: $P'(x) = 60 - 3x$. Set $P'(x) = 0 \rightarrow x = 20$.
- $P''(x) = -3 < 0$, so $x = 20$ is a maximum.
- $P(20) = 60(20) - 1.5(400) - 500 = 1200 - 600 - 500 = 100$.
- Maximum profit is $100 at a production level of 20 units.

9. Answer: r = 3 in and h = 6 in; minimum surface area = 54π sq in

- From the volume constraint: $\pi r^2h = 54\pi \rightarrow h = 54/r^2$.
- Total surface area: $S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r(54/r^2) = 2\pi r^2 + 108\pi/r$.
- Differentiate: $S'(r) = 4\pi r - 108\pi/r^2$. Set $S'(r) = 0 \rightarrow 4\pi r = 108\pi/r^2 \rightarrow r^3 = 27 \rightarrow r = 3$.
- Then $h = 54/9 = 6$.
- $S''(r) = 4\pi + 216\pi/r^3 > 0$ → confirms minimum.
- Minimum surface area = $2\pi(9) + 2\pi(3)(6) = 18\pi + 36\pi = 54\pi$ sq in.

10. Answer: r = 12/(4 + π) ≈ 1.68 m; h = 6/(4 + π) ≈ 0.84 m

- The window width equals $2r$ (diameter of semicircle). Perimeter: $2h + 2r + \pi r = 12$, so $h = (12 - 2r - \pi r)/2 = 6 - r - (\pi/2)r$.
- Total area: $A = 2rh + (1/2)\pi r^2 = 2r[6 - r - (\pi/2)r] + (\pi/2)r^2 = 12r - 2r^2 - \pi r^2 + (\pi/2)r^2$.
- Simplify: $A(r) = 12r - 2r^2 - (\pi/2)r^2$.
- Differentiate: $A'(r) = 12 - 4r - \pi r = 12 - r(4 + \pi)$. Set $A'(r) = 0 \rightarrow r = 12/(4 + \pi)$.
- $A''(r) = -(4 + \pi) < 0$ → confirms maximum.
- $h = 6 - r - (\pi/2)r = 6 - r(1 + \pi/2) = 6 - [12/(4+\pi)] \cdot (2+\pi)/2 = 6/(4+\pi) \approx 0.84$ m.

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