# Riemann Sums: Approximating Area Under a Curve

Calculus Worksheet · Grade 11–12 / College Intro Calculus

Name: ______________________________ Date: __________________

**Learning Objectives**

- Approximate the area under a curve using Left Riemann Sums (L_n), Right Riemann Sums (R_n), and Midpoint Riemann Sums (M_n)
- Calculate the width of each subinterval and evaluate the function at the correct sample points
- Compare overestimates and underestimates produced by different Riemann Sum methods for increasing and decreasing functions

## Problems

**1.** Find the width of each subinterval (delta x) when approximating the area under a curve on the interval [0, 4] using 4 equal subintervals.

$$\Delta x = \frac{b-a}{n}$$

**2.** Using the Left Riemann Sum with 4 rectangles (L4), list the four left-endpoint x-values on the interval [0, 4] where the heights of the rectangles are evaluated.

$$f(x) = x^2 + 1, \quad [0, 4], \quad n = 4$$

**3.** Evaluate the function at each of the four left endpoints and find the height of each rectangle for the Left Riemann Sum approximation.

| x | $f(x) = x^2 + 1$ |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |

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**4.** Using the heights found in Problem 3 and a width of 1 for each rectangle, calculate the Left Riemann Sum L4 to approximate the area under the curve on [0, 4].

$$f(x) = x^2 + 1, \quad L_4 = \Delta x[f(x_0) + f(x_1) + f(x_2) + f(x_3)]$$

**5.** Using the Right Riemann Sum with 4 rectangles (R4), identify the four right-endpoint x-values on [0, 4], evaluate f(x) at each, and compute R4.

$$f(x) = x^2 + 1, \quad R_4 = \Delta x[f(x_1) + f(x_2) + f(x_3) + f(x_4)]$$

**6.** Compare L4 and R4 for f(x) = $x^2$ + 1 on [0, 4]. State which is an overestimate and which is an underestimate, and explain why in terms of the shape of the function.

$$f(x) = x^2 + 1 \text{ is increasing on } [0, 4]$$

**7.** Using the Midpoint Riemann Sum with 4 rectangles (M4) on [0, 4], identify the four midpoint x-values, evaluate f(x) at each, and compute M4.

**8.** Approximate the area under the curve on the interval [1, 3] using the Left Riemann Sum with 4 subintervals.

$$f(x) = x^2 + 2x, \quad [1, 3], \quad n = 4$$

**9.** Approximate the area under the curve on [0, 3] using the Right Riemann Sum with 6 equal subintervals.

$$f(x) = \sqrt{x + 1}, \quad [0, 3], \quad n = 6$$

**10.** The exact area under f(x) = $x^2$ + 1 on [0, 4] found using a definite integral is 76/3 ≈ 25.33 square units. Using your results from L4, R4, and M4, calculate the absolute error for each method and determine which Riemann Sum gives the best approximation.

| Method | Approximation | Exact Value | Absolute Error |
|---|---|---|---|
| L4 | 18 | 25.33 | |
| R4 | 34 | 25.33 | |

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| Method | Approximation | Exact Value | Absolute Error |
| --- | --- | --- | --- |
| M4 | 25 | 25.33 | |



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# Riemann Sums: Approximating Area Under a Curve — Answer Key

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## Answer Key

**1. Answer: Δx = 1**

- Identify: a = 0, b = 4, n = 4
- Δx = (4 – 0) / 4 = 4 / 4 = 1

**2. Answer: x = 0, 1, 2, 3**

- Δx = (4 – 0) / 4 = 1
- Left endpoints start at a = 0 and increase by Δx: x■ = 0, x■ = 1, x■ = 2, x■ = 3

**3. Answer: f(0)=1, f(1)=2, f(2)=5, f(3)=10**

| x | $f(x) = x^2 + 1$ |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 2 | 5 |
| 3 | 10 |

- $f(0) = 0^2 + 1 = 1$
- $f(1) = 1^2 + 1 = 2$
- $f(2) = 2^2 + 1 = 5$
- $f(3) = 3^2 + 1 = 10$

**4. Answer: L4 = 18 square units**

- Δx = 1, heights: f(0)=1, f(1)=2, f(2)=5, f(3)=10
- L4 = 1 × (1 + 2 + 5 + 10) = 1 × 18 = 18

**5. Answer: R4 = 34 square units**

- Δx = 1, right endpoints: x = 1, 2, 3, 4
- f(1)=2, f(2)=5, f(3)=10, f(4)=17
- R4 = 1 × (2 + 5 + 10 + 17) = 34

**6. Answer: L4 = 18 (underestimate); R4 = 34 (overestimate); because f is increasing.**

- $f(x) = x^2 + 1$ is strictly increasing on [0, 4]
- For an increasing function, left endpoints give rectangle heights below the curve → underestimate: L4 = 18
- Right endpoints give rectangle heights above the curve → overestimate: R4 = 34

**7. Answer: M4 = 25 square units**

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- Δx = 1; midpoints of [0,1],[1,2],[2,3],[3,4] are x = 0.5, 1.5, 2.5, 3.5
- f(0.5) = 0.25+1 = 1.25; f(1.5) = 2.25+1 = 3.25; f(2.5) = 6.25+1 = 7.25; f(3.5) = 12.25+1 = 13.25
- M4 = 1 × (1.25 + 3.25 + 7.25 + 13.25) = 25

### 8. Answer: L4 = 14.75 square units

- Δx = (3–1)/4 = 0.5; left endpoints: x = 1, 1.5, 2, 2.5
- f(1) = 1+2 = 3; f(1.5) = 2.25+3 = 5.25; f(2) = 4+4 = 8; f(2.5) = 6.25+5 = 11.25
- L4 = 0.5 × (3 + 5.25 + 8 + 11.25) = 0.5 × 27.5 = 13.75
- Note: L4 = 13.75 square units

### 9. Answer: R6 ≈ 4.74 square units

- Δx = (3–0)/6 = 0.5; right endpoints: x = 0.5, 1, 1.5, 2, 2.5, 3
- f(0.5)=√1.5≈1.225; f(1)=√2≈1.414; f(1.5)=√2.5≈1.581; f(2)=√3≈1.732; f(2.5)=√3.5≈1.871; f(3)=√4=2
- Sum ≈ 1.225+1.414+1.581+1.732+1.871+2 = 9.823
- R6 = 0.5 × 9.823 ≈ 4.91 square units

### 10. Answer: Errors: L4≈7.33, R4≈8.67, M4≈0.33; M4 is the best approximation.

| Method | Approximation | Exact Value | Absolute Error |
|---|---|---|---|
| L4 | 18 | 25.33 | 7.33 |
| R4 | 34 | 25.33 | 8.67 |
| M4 | 25 | 25.33 | 0.33 |

- Exact area = 76/3 ≈ 25.33
- |L4 – Exact| = |18 – 25.33| = 7.33
- |R4 – Exact| = |34 – 25.33| = 8.67
- |M4 – Exact| = |25 – 25.33| = 0.33
- M4 has the smallest absolute error (0.33), confirming it is the most accurate of the three Riemann Sum methods.

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