Volume of Solids with Known Cross Sections

Calculus Worksheet · Grade 11-12

Name: ______________________________ Date: _______________ Score: / 10

Learning Objectives

- Set up integrals to compute the volume of a solid with known cross sections
- Express the side length of a cross section using the base region's equation
- Evaluate definite integrals to find exact volumes of solids

For each solid, set up the volume integral using the given base and cross-section shape, then evaluate.

1. The base of a solid is the region inside the circle x^2 + y^2 = 4. Cross sections perpendicular to the x-axis are squares. Find the volume.

$$V = \int_{-2}^{2} \left(2\sqrt{4 - x^2}\right)^2 \, dx$$

Answer: _________________________________

2. The base of a solid is the region inside the circle x^2 + y^2 = 9. Cross sections perpendicular to the x-axis are squares. Find the volume.

$$V = \int_{-3}^{3} \left(2\sqrt{9 - x^2}\right)^2 \, dx$$

Answer: _________________________________

3. The base of a solid is the region inside the circle x^2 + y^2 = 1. Cross sections perpendicular to the x-axis are squares. Find the volume.

$$V = \int_{-1}^{1} \left(2\sqrt{1 - x^2}\right)^2 \, dx$$

Answer: _________________________________

4. The base of a solid is the region inside the circle x^2 + y^2 = 4. Cross sections perpendicular to the x-axis are semicircles. Find the volume.

$$V = \int_{-2}^{2} \frac{\pi}{2} \left(\sqrt{4 - x^2}\right)^2 \, dx$$

Answer: _________________________________

5. The base of a solid is the region inside the circle x^2 + y^2 = 4. Cross sections perpendicular to the x-axis are equilateral triangles. Find the volume.

$$V = \int_{-2}^{2} \frac{\sqrt{3}}{4} \left(2\sqrt{4 - x^2}\right)^2 \, dx$$

Answer: _________________________________

6. The base of a solid is the region bounded by y = x^2 and y = 4. Cross sections perpendicular to the y-axis are squares. Find the volume.

$$V = \int_0^4 (2\sqrt{y})^2 \, dy$$

Answer: ______________________________

7. The base of a solid is the region bounded by y = x and y = x^2. Cross sections perpendicular to the x-axis are squares. Find the volume.

$$V = \int_0^1 (x - x^2)^2 \, dx$$

Answer: ______________________________

8. The base of a solid is the region inside the ellipse x^2/9 + y^2/4 = 1. Cross sections perpendicular to the x-axis are squares. Find the volume.

$$V = \int_{-3}^{3} \left(\frac{4}{3}\sqrt{9 - x^2}\right)^2 \, dx$$

Answer: ______________________________

9. The base of a solid is the region inside the circle x^2 + y^2 = 16. Cross sections perpendicular to the x-axis are isosceles right triangles with the hypotenuse on the base. Find the volume.

$$V = \int_{-4}^{4} \frac{1}{4}\left(2\sqrt{16 - x^2}\right)^2 \, dx$$

Answer: ______________________________

10. The base of a solid is the region bounded by y = sqrt(x), y = 0, and x = 4. Cross sections perpendicular to the x-axis are squares. Find the volume.

$$V = \int_0^4 (\sqrt{x})^2 \, dx$$

Answer: ______________________________

Encourage students to sketch the base region and a representative cross section before writing the integral.

Solutions

1. The base of a solid is the region inside the circle x^2 + y^2 = 4. Cross sections perpendicular to the x-axis are squares. Find the volume.

$$V = \int_{-2}^{2} (2\sqrt{4 - x^2})^2 \, dx$$

→ *Solve the circle equation for y to get y equals plus or minus the square root of 4 minus x squared*
→ *The side of the square equals the full vertical chord, which is 2 times the square root of 4 minus x squared*
→ *Square the side to get the cross-sectional area, then integrate from negative 2 to 2*
→ *Evaluate the integral to obtain 128 divided by 3*

Answer: $\frac{128}{3}$

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2. The base of a solid is the region inside the circle x^2 + y^2 = 9. Cross sections perpendicular to the x-axis are squares. Find the volume.

$$V = \int_{-3}^{3} (2\sqrt{9 - x^2})^2 \, dx$$

→ *Solve for y to get y equals plus or minus the square root of 9 minus x squared*
→ *The square's side is 2 times the square root of 9 minus x squared*
→ *Square the side and integrate from negative 3 to 3*
→ *Evaluate to obtain 144*

Answer: 144

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3. The base of a solid is the region inside the circle x^2 + y^2 = 1. Cross sections perpendicular to the x-axis are squares. Find the volume.

$$V = \int_{-1}^{1} (2\sqrt{1 - x^2})^2 \, dx$$

→ *Solve for y to get y equals plus or minus the square root of 1 minus x squared*
→ *The side of the square is 2 times the square root of 1 minus x squared*
→ *Square the side and integrate from negative 1 to 1*
→ *Evaluate to obtain 16 divided by 3*

Answer: $\frac{16}{3}$

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4. The base of a solid is the region inside the circle x^2 + y^2 = 4. Cross sections perpendicular to the x-axis are semicircles. Find the volume.

$$V = \int_{-2}^{2} \frac{\pi}{2} (\sqrt{4 - x^2})^2 \, dx$$

→ *The diameter of each semicircle equals the vertical chord, which is 2 times the square root of 4 minus x squared*
→ *The radius is half of that, equal to the square root of 4 minus x squared*
→ *The semicircle area is one half times pi times the radius squared*
→ *Integrate from negative 2 to 2 to get 16 pi divided by 3*

Answer: $\frac{16\pi}{3}$

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5. The base of a solid is the region inside the circle x^2 + y^2 = 4. Cross sections perpendicular to the x-axis are equilateral triangles. Find the volume.

$$V = \int_{-2}^{2} \frac{\sqrt{3}}{4}\left(2\sqrt{4-x^2}\right)^2 dx$$

→ *The side of the equilateral triangle equals the full chord, which is 2 times the square root of 4 minus x squared*
→ *The area of an equilateral triangle is the square root of 3 over 4 times the side squared*
→ *Set up and integrate from negative 2 to 2*
→ *Evaluate to obtain 32 times the square root of 3 divided by 3*

Answer: $\frac{32\sqrt{3}}{3}$

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6. The base of a solid is the region bounded by y = x^2 and y = 4. Cross sections perpendicular to the y-axis are squares. Find the volume.

$$V = \int_{0}^{4} \left(2\sqrt{y}\right)^2 dy$$

→ *Solve y equals x squared for x to get x equals plus or minus the square root of y*
→ *The square has side length 2 times the square root of y*
→ *Square the side and integrate from 0 to 4*
→ *Evaluate to obtain 32*

Answer: 32

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7. The base of a solid is the region bounded by y = x and y = x^2. Cross sections perpendicular to the x-axis are squares. Find the volume.

$$V = \int_{0}^{1} (x - x^2)^2 dx$$

→ *Find the intersections of y equals x and y equals x squared, which are at x equals 0 and x equals 1*
→ *The side of the square is the top function minus the bottom function, x minus x squared*
→ *Square the side and integrate from 0 to 1*
→ *Evaluate to obtain 1 divided by 30*

Answer: $\frac{1}{30}$

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8. The base of a solid is the region inside the ellipse x^2/9 + y^2/4 = 1. Cross sections perpendicular to the x-axis are squares. Find the volume.

$$V = \int_{-3}^{3} \left(\frac{4}{3}\sqrt{9-x^2}\right)^2 dx$$

→ *Solve the ellipse equation for y to get y equals plus or minus two thirds times the square root of 9 minus x squared*
→ *The side of the square is the full chord, four thirds times the square root of 9 minus x squared*
→ *Square the side and integrate from negative 3 to 3*
→ *Evaluate to obtain 64*

Answer: 64

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9. The base of a solid is the region inside the circle x^2 + y^2 = 16. Cross sections perpendicular to the x-axis are isosceles right triangles with the hypotenuse on the base. Find the volume.

$$V = \int_{-4}^{4} \frac{1}{4}\left(2\sqrt{16 - x^2}\right)^2 dx$$

→ *The hypotenuse equals the full chord, 2 times the square root of 16 minus x squared*
→ *The area of an isosceles right triangle with hypotenuse h is h squared divided by 4*
→ *Set up and integrate from negative 4 to 4*
→ *Evaluate to obtain 256 divided by 3*

Answer: $\frac{256}{3}$

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10. The base of a solid is the region bounded by y = sqrt(x), y = 0, and x = 4. Cross sections perpendicular to the x-axis are squares. Find the volume.

$$V = \int_{0}^{4} (\sqrt{x})^2 dx$$

→ *The side of each square equals the height of the region, which is the square root of x*
→ *Square the side to get x as the cross-sectional area*
→ *Integrate x from 0 to 4*
→ *Evaluate to obtain 8*

Answer: 8