Volumes of Solids of Revolution: The Washer Method

Calculus Worksheet · Grade 11-12

Name: ______________________________ Date: _______________ Score: / 10

Learning Objectives

- Identify outer and inner radii for solids of revolution with a hole in the middle
- Set up definite integrals using the washer method formula
- Compute volumes of solids generated by rotating regions about horizontal and vertical axes

For each problem, sketch the region, identify the outer and inner radii, set up the washer-method integral, and evaluate to find the exact volume.

1. Find the volume of the solid generated by revolving the region bounded by y = x and y = x^2 about the x-axis.

$$V = \pi \int_0^1 (x^2 - (x^2)^2)\, dx$$

Answer: ______________________________

2. Find the volume of the solid generated by revolving the region bounded by y = x and y = x^2 about the line y = -1.

$$V = \pi \int_0^1 ((x+1)^2 - (x^2+1)^2)\, dx$$

Answer: ______________________________

3. Find the volume of the solid generated by revolving the region bounded by y = sqrt(x) and y = x^2 about the x-axis.

$$V = \pi \int_0^1 (x - x^4)\, dx$$

Answer: ______________________________

4. Find the volume of the solid generated by revolving the region bounded by y = 2x and y = x^2 about the x-axis.

$$V = \pi \int_0^2 (4x^2 - x^4)\, dx$$

Answer: ______________________________

5. Find the volume of the solid generated by revolving the region bounded by y = x^2 and y = 4 about the x-axis.

$$V = \pi \int_{-2}^2 (16 - x^4)\, dx$$

Answer: ______________________________

6. Find the volume of the solid generated by revolving the region bounded by y = x and y = x^2 about the line y = 2.

$$V = \pi\int_0^1 ((2-x^2)^2 - (2-x)^2)\,dx$$

Answer: ______________________________

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7. Find the volume of the solid generated by revolving the region bounded by y = x^2 and y = 2x about the line y = 5.

$$V = \pi\int_0^2 ((5-x^2)^2 - (5-2x)^2)\,dx$$

Answer: ______________________________

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8. Find the volume of the solid generated by revolving the region bounded by y = x^2 and y = x about the line y = -2.

$$V = \pi\int_0^1 ((x+2)^2 - (x^2+2)^2)\,dx$$

Answer: ______________________________

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9. Find the volume of the solid generated by revolving the region bounded by y = sqrt(x), y = 0, and x = 4 about the line y = 3.

$$V = \pi\int_0^4 (9 - (3-\sqrt{x})^2)\,dx$$

Answer: ______________________________

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10. Find the volume of the solid generated by revolving the region bounded by y = x^3 and y = x about the x-axis in the first quadrant.

$$V = \pi\int_0^1 (x^2 - x^6)\,dx$$

Answer: ______________________________

Encourage students to always sketch the region and the axis of rotation before writing the integral; emphasize that the washer formula requires squaring each radius separately, not subtracting first.

Solutions

1. Find the volume of the solid generated by revolving the region bounded by y = x and y = x^2 about the x-axis.

$$V = \pi\int_0^1 (x^2 - (x^2)^2)\,dx$$

- → *Find the intersection points by setting x equal to x squared, giving x equals 0 and x equals 1.*
- → *The outer radius is the line y equals x and the inner radius is the parabola y equals x squared.*
- → *Set up the washer integral with pi times the integral from 0 to 1 of x squared minus x to the fourth dx.*
- → *Integrate term by term to get x cubed over 3 minus x to the fifth over 5 evaluated from 0 to 1.*
- → *Substitute the limits to get one third minus one fifth, which equals two fifteenths.*
- → *Multiply by pi to obtain the final volume of two pi over fifteen.*

Answer: $V = \frac{2\pi}{15}$

2. Find the volume of the solid generated by revolving the region bounded by y = x and y = x^2 about the line y = -1.

$$V = \pi\int_0^1 ((x+1)^2 - (x^2+1)^2)\,dx$$

- → *Shift both functions up by 1 to account for the axis of rotation at y equals negative 1.*
- → *The outer radius is x plus 1 and the inner radius is x squared plus 1.*
- → *Expand the squares to get x squared plus 2x plus 1 minus x to the fourth plus 2x squared plus 1.*
- → *Simplify the integrand to 2x minus x squared minus x to the fourth.*
- → *Integrate from 0 to 1 to get x squared minus x cubed over 3 minus x to the fifth over 5.*
- → *Evaluate to get 1 minus one third minus one fifth, which equals twenty-nine over thirty, then multiply by pi.*

Answer: $V = \frac{29\pi}{30}$

3. Find the volume of the solid generated by revolving the region bounded by y = sqrt(x) and y = x^2 about the x-axis.

$$V = \pi\int_0^1 (x - x^4)\,dx$$

- → *Find the intersection points by setting square root of x equal to x squared, giving x equals 0 and x equals 1.*
- → *The outer radius is square root of x and the inner radius is x squared.*
- → *Square each radius to obtain x and x to the fourth respectively.*
- → *Set up the integral pi times the integral from 0 to 1 of x minus x to the fourth dx.*
- → *Integrate to get x squared over 2 minus x to the fifth over 5 evaluated from 0 to 1.*
- → *Compute one half minus one fifth to obtain three tenths, then multiply by pi.*

Answer: $V = \frac{3\pi}{10}$

4. Find the volume of the solid generated by revolving the region bounded by y = 2x and y = x^2 about the x-axis.

$$V = \pi \int_0^2 (4x^2 - x^4)\, dx$$

→ *Find intersections by setting 2x equal to x squared, giving x equals 0 and x equals 2.*
→ *The outer radius is 2x and the inner radius is x squared.*
→ *Square the radii to get 4x squared and x to the fourth.*
→ *Integrate pi times the integral from 0 to 2 of 4x squared minus x to the fourth dx.*
→ *Evaluate the antiderivative four x cubed over 3 minus x to the fifth over 5 at the bounds.*
→ *Simplify to thirty-two thirds minus thirty-two fifths, giving sixty-four fifteenths times pi.*

Answer: $V = \frac{64\pi}{15}$

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5. Find the volume of the solid generated by revolving the region bounded by y = x^2 and y = 4 about the x-axis.

$$V = \pi \int_{-2}^2 (16 - x^4)\, dx$$

→ *Determine the bounds by setting x squared equal to 4, giving x equals negative 2 and x equals 2.*
→ *The outer radius is the constant 4 and the inner radius is x squared.*
→ *Square each radius to get 16 and x to the fourth.*
→ *Integrate pi times the integral from negative 2 to 2 of 16 minus x to the fourth dx.*
→ *Compute the antiderivative 16x minus x to the fifth over 5 at the limits.*
→ *Simplify to get 256 pi over 5 as the final volume.*

Answer: $V = \frac{256\pi}{5}$

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6. Find the volume of the solid generated by revolving the region bounded by y = x and y = x^2 about the line y = 2.

$$V = \pi \int_0^1 \left((2 - x^2)^2 - (2 - x)^2\right) dx$$

→ *Since the axis y equals 2 is above the region, the outer radius is 2 minus x squared and the inner radius is 2 minus x.*
→ *Expand each squared expression carefully to get 4 minus 4x squared plus x to the fourth minus the expansion of 2 minus x squared.*
→ *Simplify the integrand to x to the fourth minus 4x squared minus x squared plus 4x, then combine like terms.*
→ *Integrate the simplified polynomial from 0 to 1.*
→ *Evaluate the antiderivative at the bounds.*
→ *Multiply by pi to obtain eight pi over fifteen.*

Answer: $V = \frac{8\pi}{15}$

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7. Find the volume of the solid generated by revolving the region bounded by y = x^2 and y = 2x about the line y = 5.

$$V = \pi \int_0^2 \left((5 - x^2)^2 - (5 - 2x)^2\right) dx$$

→ *Determine intersection points where x squared equals 2x, giving x equals 0 and x equals 2.*
→ *Because y equals 5 lies above the region, outer radius is 5 minus x squared and inner radius is 5 minus 2x.*
→ *Expand both squared terms and subtract carefully.*
→ *Combine like terms to obtain a polynomial integrand.*
→ *Integrate from 0 to 2 and evaluate at the bounds.*
→ *Multiply the result by pi to get one hundred thirty-six pi over fifteen.*

Answer: $V = \frac{136\pi}{15}$

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8. Find the volume of the solid generated by revolving the region bounded by y = x^2 and y = x about the line y = -2.

$$V = \pi \int_0^1 \left((x+2)^2 - (x^2+2)^2\right) dx$$

- → *Shift each function by adding 2 since the axis of rotation is y equals negative 2.*
- → *The outer radius is x plus 2 and the inner radius is x squared plus 2.*
- → *Expand each squared radius to get x squared plus 4x plus 4 and x to the fourth plus 4x squared plus 4.*
- → *Subtract to obtain the integrand 4x minus 3x squared minus x to the fourth.*
- → *Integrate from 0 to 1 to get 2x squared minus x cubed minus x to the fifth over 5.*
- → *Evaluate to obtain seventeen fifteenths and multiply by pi.*

Answer: $V = \frac{17\pi}{15}$

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9. Find the volume of the solid generated by revolving the region bounded by y = sqrt(x), y = 0, and x = 4 about the line y = 3.

$$V = \pi \int_0^4 \left(9 - (3 - \sqrt{x})^2\right) dx$$

- → *Identify the outer radius as 3 (distance from x-axis to y equals 3) and the inner radius as 3 minus square root of x.*
- → *Square the radii to get 9 and 9 minus 6 square root of x plus x.*
- → *Subtract to simplify the integrand to 6 square root of x minus x.*
- → *Integrate from 0 to 4, giving 4x raised to the three halves minus x squared over 2.*
- → *Evaluate the antiderivative at x equals 4.*
- → *Multiply by pi to obtain forty pi over three.*

Answer: $V = \frac{40\pi}{3}$

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10. Find the volume of the solid generated by revolving the region bounded by y = x^3 and y = x about the x-axis in the first quadrant.

$$V = \pi \int_0^1 (x^2 - x^6)\, dx$$

- → *Find intersections in the first quadrant by setting x cubed equal to x, giving x equals 0 and x equals 1.*
- → *Outer radius is the line y equals x and inner radius is the curve y equals x cubed.*
- → *Square the radii to obtain x squared and x to the sixth.*
- → *Set up the integral pi times the integral from 0 to 1 of x squared minus x to the sixth dx.*
- → *Integrate to get x cubed over 3 minus x to the seventh over 7.*
- → *Evaluate at the bounds to find one third minus one seventh, equaling four twenty-firsts, and multiply by pi.*

Answer: $V = \frac{4\pi}{21}$