Mathematical Induction in Precalculus

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Learning Objectives

- Identify the three steps of mathematical induction (base case, assume n=k, prove n=k+1)
- Verify base cases for summation and divisibility statements
- Use the inductive hypothesis and the FOIL method to simplify the n=k+1 step
- Write complete induction proofs for standard summation formulas

For each problem, use the principle of mathematical induction to prove the given statement, showing the base case, inductive hypothesis, and inductive step.

1. State the three steps required to prove a statement by mathematical induction.

List the three steps of mathematical induction.

Answer: ______________________

2. Verify the base case n=1 for the formula shown.

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Answer: ______________________

3. Write the inductive hypothesis (the n=k statement) for the sum-of-first-n formula.

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Answer: ______________________

4. Write the statement that must be proved for n=k+1 for the sum-of-first-n formula.

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Answer: ______________________

5. Use the FOIL method to expand the expression that appears in the induction step.

$$(k+1)(k+2)$$

Answer: ______________________

6. Complete the inductive step: starting from the inductive hypothesis, prove the formula for n=k+1.

$$\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

Answer: ______________________

7. Verify the base case n=1 for the sum of the first n odd numbers.

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Answer: ______________________________________

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8. Prove the inductive step for the sum of the first n odd numbers using the hypothesis 1+3+5+...+(2k-1)=k^2.

$$k^2 + (2k + 1) = (k + 1)^2$$

Answer: ______________________________________

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9. Verify the base case n=1 for the sum-of-squares formula.

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$$

Answer: ______________________________________

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10. Use induction to verify the base case and write the n=k+1 statement for the divisibility claim that 3 divides 4^n - 1.

$$3 \text{ divides } 4^n - 1$$

Answer: ______________________________________

This worksheet covers the method of mathematical induction as introduced in the video: (1) proving the base case n=1, (2) assuming the statement is true for n=k (inductive hypothesis), and (3) proving the statement holds for n=k+1, including use of the FOIL method to expand and simplify. Problems include the featured proof 1+2+3+...+n = n(n+1)/2 plus other standard summation and divisibility induction proofs.

Solutions

1. State the three steps required to prove a statement by mathematical induction.

List the three steps of mathematical induction.

→ *The first step is the base case: show the statement holds when n equals 1.*
→ *The second step is the inductive hypothesis: assume the statement is true when n equals k.*
→ *The third step is the inductive step: using the assumption, prove the statement is true when n equals k plus 1.*

Answer: (1) Prove true for $n=1$, (2) Assume true for $n=k$, (3) Prove true for $n=k+1$

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2. Verify the base case n=1 for the formula shown.

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

→ *Substitute n equal to 1 into the left side to get 1.*
→ *Substitute n equal to 1 into the right side to get one times two divided by two, which equals 1.*
→ *Since both sides equal 1, the base case is verified.*

Answer: $$1=\frac{1(1+1)}{2}=1$$

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3. Write the inductive hypothesis (the n=k statement) for the sum-of-first-n formula.

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

→ *Replace every n on the left side with k to get the sum 1 plus 2 plus 3 and so on up to k.*
→ *Replace every n on the right side with k to get k times the quantity k plus 1, all divided by 2.*
→ *This equation is assumed true and is called the inductive hypothesis.*

Answer: $$1+2+3+\cdots+k=\frac{k(k+1)}{2}$$

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4. Write the statement that must be proved for n=k+1 for the sum-of-first-n formula.

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

→ *Replace n with k plus 1 on the left side so the last term added is k plus 1.*
→ *Replace n with k plus 1 on the right side to get the quantity k plus 1 times the quantity k plus 2, divided by 2.*
→ *This is the equation that must be shown to hold in the inductive step.*

Answer: $$1+2+\cdots+k+(k+1)=\frac{(k+1)(k+2)}{2}$$

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5. Use the FOIL method to expand the expression that appears in the induction step.

$(k+1)(k+2)$

→ *Multiply the first terms k and k to get k squared.*
→ *Multiply the outer terms k and 2 to get 2k, and the inner terms 1 and k to get k.*
→ *Multiply the last terms 1 and 2 to get 2, then combine like terms to get k squared plus 3k plus 2.*

Answer: $k^2 + 3k + 2$

6. Complete the inductive step: starting from the inductive hypothesis, prove the formula for n=k+1.

$\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$

→ *Add k plus 1 to both sides of the inductive hypothesis to introduce the next term in the sum.*
→ *Combine the right side over a common denominator of 2 to get k times k plus 1 plus 2 times k plus 1, all over 2.*
→ *Factor k plus 1 from the numerator to obtain k plus 1 times k plus 2 divided by 2, which matches the n=k+1 statement.*

Answer: $\frac{(k+1)(k+2)}{2}$

7. Verify the base case n=1 for the sum of the first n odd numbers.

$1 + 3 + 5 + \cdots + (2n-1) = n^2$

→ *Substitute n equal to 1 on the left side to get 2 times 1 minus 1, which equals 1.*
→ *Substitute n equal to 1 on the right side to get 1 squared, which equals 1.*
→ *Both sides equal 1, so the base case is verified.*

Answer: $1 = 1^2 = 1$

8. Prove the inductive step for the sum of the first n odd numbers using the hypothesis 1+3+5+...+(2k-1)=k^2.

$k^2 + (2k+1) = (k+1)^2$

→ *Add the next odd term 2k plus 1 to both sides of the inductive hypothesis.*
→ *Combine the right side to get k squared plus 2k plus 1.*
→ *Recognize this as the perfect square k plus 1 squared, completing the inductive step.*

Answer: $(k+1)^2$

9. Verify the base case n=1 for the sum-of-squares formula.

$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

→ *Substitute n equal to 1 on the left side to get 1 squared, which equals 1.*
→ *Substitute n equal to 1 on the right side to get 1 times 2 times 3 divided by 6, which equals 1.*
→ *Both sides equal 1, so the base case holds.*

Answer: $1 = \frac{1 \cdot 2 \cdot 3}{6} = 1$

10. Use induction to verify the base case and write the n=k+1 statement for the divisibility claim that 3 divides 4^n - 1.

3 divides $4^n - 1$

→ *For the base case n equal to 1, compute 4 to the first power minus 1 to get 3, which is divisible by 3.*
→ *Assume that 3 divides 4 to the k minus 1, the inductive hypothesis.*
→ *The statement to prove for n equal to k plus 1 is that 3 divides 4 to the power k plus 1 minus 1.*

Answer: 3 divides $4^{k+1} - 1$